Regular propagators of bilinear quantum systems

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Abstract

The present analysis deals with the regularity of solutions of bilinear control systems of the type $x' = (A + u(t)B)x$ where the state x belongs to some complex infinite dimensional Hilbert space, the (possibly unbounded) linear operators A and B are skew-adjoint and the control u is a real valued function. Such systems arise, for instance, in quantum control with the bilinear Schrödinger equation. For the sake of the regularity analysis, we consider a more general framework where A and B are generators of contraction semigroups.

Under some hypotheses on the commutator of the operators A and B, it is possible to extend the definition of solution for controls in the set of Radon measures to obtain precise *a priori* energy estimates on the solutions, leading to a natural extension of the celebrated noncontrollability result of Ball, Marsden, and Slemrod in 1982.

Index terms— Quantum Control; Bilinear Schrödinger equation

Contents

1 Introduction

A bilinear control system in a Banach space $\mathcal{X}$ is given by an evolution equation

$$\frac{d}{dt}x(t) = (A + u(t)B)x(t) \tag{1.1}$$

where A and B are two (possibly unbounded) linear operators on $\mathcal{X}$ and u is a real-valued function, the control. Well-posedness of bilinear evolution equations of type (1.1) for a given control u is usually a difficult question. If K is a subset of $\mathbf{R}$, we define $PC(K)$ the set of right-continuous piecewise constant functions taking values in K.

If K, A and B are such that for every u in K, $A + uB$ generates a C^0 semigroup $t \mapsto e^{t(A+uB)}$, then for every $T \geq 0$ and every u in $PC(K)$, the restriction of u on $[0, T)$ writes

$$u = \sum_{j=1}^{p} u_j \mathbb{I}_{[\tau_j, \tau_{j+1})} \tag{1.2}$$

with $p \in \mathbf{N}$, $u_1, \ldots, u_p \in K$ and $\tau_1 < \tau_2 < \ldots < \tau_{p+1} = T$, and one defines the associated propagator of (1.1) by

$$\Upsilon^u_{t,\tau_1} = e^{(t-\tau_j)(A+u_jB)} \circ e^{(\tau_j-\tau_{j-1})(A+u_{j-1}B)} \circ \cdots \circ e^{(\tau_2-\tau_1)(A+u_1B)},$$

for every t in (τ_j, τ_{j+1}). The solution of (1.1) with initial value x_0 at time τ_1 is $t \mapsto \Upsilon^u_{t,\tau_1}\psi_0$. When $\tau_1 = 0$, we denote $\Upsilon^u_t := \Upsilon^u_{t,0}$.

It is of particular interest in the applications to study the set of points that can be attained in finite time from a given initial datum ψ_0 using a set of admissible controls in $\mathcal{Z}$

$$\mathcal{A}tt_{\mathcal{Z}}(\psi_0) = \cup_{t \geq 0}\{\Upsilon^u_t \psi_0 | u \in \mathcal{Z}\}$$

where $\mathcal{Z}$ is a subset of $PC(K)$ or, possibly, a larger set (provided that a suitable extension of Υ to $\mathcal{Z}$ makes sense). The set $\mathcal{A}tt_{\mathcal{Z}}(\psi_0)$ is called attainable set from ψ_0 with controls $\mathcal{Z}$.

Providing a precise description of the propagator is, in principle, a hard task and, in turns, so is studying the controllability of (1.1). On the other hand, one could use the regularity of the solutions of (1.1) to provide upper bounds of the attainable sets of the bilinear system in order to determine obstructions to controllability. The present analysis focuses on this second approach.

1.1 Elementary obstructions to controllability in a Banach space

There are several upper bounds on the attainable sets that can be deduced from natural properties of the system. We list below some of them.

1.1.1 Conservation of the norm

In the Hilbertian case, in which $\mathcal{X}$ is a Hilbert space, the propagator $t \mapsto \Upsilon^u_t$ is unitary as soon as $A + uB$ is essentially skew-adjoint for every u in K. If $PC(K)$ is endowed with a topology for which $u \mapsto \Upsilon^u_T \psi_0$ is continuous for every $T > 0$ and every ψ_0 in $\mathcal{X}$, then the continuous extension of the mapping $u \mapsto \Upsilon^u_T \psi_0$ takes value in the sphere of radius $\|\psi_0\|$.

1.1.2 Continuity of the propagators

In the general case in which $\mathcal{X}$ is a Banach space, let $\mathcal{Z}$ be a topological space, containing $PC(K)$, endowed with a topology such that $PC(K)$ is dense in $\mathcal{Z}$ and $u \in \mathcal{Z} \mapsto \Upsilon^u_T \psi_0 \in \mathcal{X}$ is continuous for every $T > 0$ and every ψ_0 in $\mathcal{X}$. Assume, moreover, that $u \mapsto \Upsilon^u_T \psi_0$ admits a (necessarily unique) continuous extension to $\mathcal{Z}$. If $\mathcal{Z}_0 \subset \mathcal{Z}$, endowed with a topology finer than the one induced by $\mathcal{Z}$, is sequentially compact (for its own topology), then for every ψ_0 in $\mathcal{X}$, for every $T > 0$, the attainable set at time T from ψ_0 with controls in $\mathcal{Z}_0$, $\{\Upsilon^u_T \psi_0 | u \in \mathcal{Z}_0\}$ is compact.

If $(\mathcal{Z}_i)_{i\in\mathbf{N}}$ is a countable covering of $\mathcal{Z}$, $\mathcal{Z} = \cup_{i\in\mathbf{N}}\mathcal{Z}_i$, $\mathcal{Z}_i$ is sequentially compact for every i, and the topology of $\mathcal{Z}_i$ is finer than the topology induced by $\mathcal{Z}$, then the attainable set at time T from ψ_0 with controls in $\mathcal{Z}$, $\{\Upsilon^u_T\psi_0 | u \in \mathcal{Z}\} = \cup_{i\in\mathbf{N}}\{\Upsilon^u_T\psi_0 | u \in \mathcal{Z}_i\}$ is a countable union of compact sets in $\mathcal{X}$ (hence is a meager set in the sense of Baire as soon as $\mathcal{X}$ is infinite dimensional).

Notice that if the input-output mapping $PC(K) \ni u \mapsto \Upsilon^u\psi_0 \in C^0([0,T],\mathcal{X})$ is continuous, then the above results can be generalized to show that the attainable set from ψ_0 at time less than T $\cup_{0\le t\le T}\{\Upsilon^u_t\psi_0 | u \in \mathcal{Z}\} = \cup_{i\in\mathbf{N}} \cup_{0\le t\le T} \{\Upsilon^u_T\psi_0 | u \in \mathcal{Z}_i\}$ is an union of compact sets.

This is the underlying idea of the proof of the following result by Ball, Marsden, and Slemrod.

Theorem (Theorem 3.6 in [BMS82]). Let $\mathcal{X}$ be an infinite dimensional Banach space, A generate a C^0 semigroup of bounded linear operators on $\mathcal{X}$, and B be a bounded linear operator on $\mathcal{X}$. Then for any $T \ge 0$, the input-output mapping $u \mapsto \Upsilon^u_T$ admits a unique continuous extension to $L^1([0,T],\mathbf{R})$ and the attainable set

$$\bigcup_{r>1}\bigcup_{T\ge0}\bigcup_{u\in L^r([0,T],\mathbf{R})} \{\Upsilon^u_t\psi_0 \mid t \in [0,T]\} \tag{1.3}$$

is contained in a countable union of compact subsets of $\mathcal{X}$, and, in particular, has dense complement.

In this case, for any $T \ge 0$, $\mathcal{Z} = \cup_{r>1}L^r([0,T],\mathbf{R})$, $\mathcal{Z} = \cup_{i,j}\mathcal{Z}_{ij}$ with

$$\mathcal{Z}_{i,j} = \{f \in L^{1+\frac{1}{j}}([0,T],\mathbf{R}) \mid \|f\|_{L^{1+\frac{1}{j}}([0,T])} \le i\},$$

and $L^{1+\frac{1}{j}}([0,T],\mathbf{R})$ is endowed with the weak L^1-topology. The sequential-compactness of $\mathcal{Z}_{i,j}$ is granted by Banach–Alaoglu–Bourbaki Theorem. The main difficulty of Theorem 3.6 in [BMS82] is to prove that, for any ψ_0 in $\mathcal{X}$, the weak convergence of (u_n) to u in L^1 implies strong convergence of the associated sequence of solutions of (1.1) ($t \mapsto \Upsilon^{u_n}_t\psi_0$) to $t \mapsto \Upsilon^u_t\psi_0$.

Remark 1. The above argument does not hold anymore if one considers controls in L^1, since L^1 is not a reflexive space. This is the content of [BMS82, Remark 3.8], where the question of possible extensions of the above result to $r = 1$ is left open except in the so-called (see [Sle84]) diagonal case, see [BMS82, Theorem 5.5].

Another example of the same obstruction is given below in Corollary 10 with $\mathcal{Z}$ equal to the set of functions with bounded variations. In this case, the sequential compactness in $\mathcal{Z}$ is given by Helly's selection theorem.

1.1.3 Invariance of the domain

In the case in which A and B are bounded operators on $\mathcal{X}$, if $\mathcal{F}$ is a closed subspace of $\mathcal{X}$ left invariant by $A+uB$ for every u in K, then for every u, the C^0 semigroup generated by $A+uB$ leaves $\mathcal{F}$ invariant. Thus, for every u in $PC(K)$ and every $t \ge 0$, Υ^u_t leaves $\mathcal{F}$ invariant. If, moreover, the dynamics is time-reversible, then for every ψ_0 in $\mathcal{X}$, for every u in $PC(K)$, for every $t > 0$, $\Upsilon^u_t\psi_0 \in \mathcal{F}$ if and only if $\psi_0 \in \mathcal{F}$.

Even in the unbounded case, the same conclusion holds if $\mathcal{F}$ a subspace of $\mathcal{X}$ left invariant by the dynamics Υ^u_t and its time-reverse dynamics (when it makes sense).

We will see in Section 4.2 below that these invariance properties remain true in the Hilbert case when $\mathcal{F}$ is the domain of a power of A left invariant by B.

1.2 Attainable sets in quantum control

The main motivation for the present analysis comes from the problem of controllability for closed quantum systems. The state of a quantum system evolving on a finite dimensional Riemannian manifold Ω, with associated measure μ, is described by its *wave function*, represented as a point in the unit sphere of $L^2(\Omega, \mathbf{C})$. In absence of interactions with the environment and neglecting the relativistic effects, the time evolution of the wave function is given by the Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(x)\psi(x,t),$$

where Δ is the Laplace-Beltrami operator on Ω and $V : \Omega \to \mathbf{R}$ is a real function (usually called potential) accounting for the physical properties of the system. When submitted to an excitation by an external field (e.g. a laser), the Schrödinger equation reads

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(x)\psi(x,t) + u(t)W(x)\psi(x,t), \tag{1.4}$$

where $W : \Omega \to \mathbf{R}$ is a real function accounting for the physical properties of the external field and u is a real function of time accounting for the intensity of it.

In the last decades, many efforts have been made to describe the attainable set of (1.4). In [Tur00], Turinici adapted the result by Ball, Marsden and Slemrod [BMS82, Theorem 3.6] to (1.4) with a measurable bounded W. The first known positive result has been obtained by Beauchard in [Bea05], and improved in [BL10, BM14a], for $\Omega = (0, \pi)$ with Dirichlet boundary conditions, $V = 0$ and $W : x \mapsto x^2$, see Section 6.2 for more details. In the case of the quantum harmonic oscillator: $\Omega = \mathbf{R}$, $V(x) = x^2$ and $W : x \mapsto x$, the attainable set is finite dimensional due to symmetries of the system, see Rouchon and Mirrahimi in [MR04] and Section 6.3.

At present, very few results providing a precise description of the attainable sets of (1.4) in the spirit of [Bea05] have been obtained, and only with strong restrictions on the dimension and the boundedness of the domain Ω. Instead of lower bounds of the attainable sets, many works have considered lower bounds of its closure (in different natural norms) which is sufficient from a physical point of view. We can, for instance, cite the work by Nersesyan [Ner09], in Sobolev spaces by means of Lyapunov technics for bounded domains and potentials. Concerning unbounded domains but with bounded potentials, we can cite [Mir09] with Lyapunov technics as well. Geometric methods have been used to prove the density of the attainable set in L^2-norm when the spectrum is purely discrete and nonresonance conditions are satisfied, see [CMSB09, MS10, BCCS12, Cha12, BCS14, CS18]. The common strategy used in the above mentioned approximate controllability results is to fix an initial condition and to design a sequence of controls such that the associated trajectories converge, in some appropriate sense, to the target. If bounds were known, both on the controllability time and on some L^p norms, for these sequences of control, one could extract a convergent subsequence of controls and use the continuity of the endpoint mapping to infer exact controllability. The present work, similarly to [Ner09], considers the question of the regularity of a solution of (1.1) but in a more general way, following the spirit of [BCC13].

1.3 Impulsive control

As mentioned in Remark 1, a major difficulty in extending the result of obstruction of Ball-Marsden-Slemrod is the lack of reflectiveness of L^1 and the fact that a bounded sequence in L^1 does not necessarily admits a weakly-convergent subsequence. On the other hand, the set of Radon measures, when endowed with the total variation topology (see Appendix A), possesses weak sequential compactness properties. As a consequence, throughout this work we will deduce properties for integrable controls from the equivalent statement set in the framework of Radon measures controls.

The idea to consider measures (instead of functions) as controls has given rise to a large literature. Let us cite, among many others contributions, [Sus76], [Mil76], [DMR91], [BR88], [Bre96], [Bre08] , [BM14b]. For an introduction to the subject we refer to the book [MR03]. Nowadays, the question of the definition of solution for dynamics such as (1.1) in finite dimensional spaces is essentially well-understood. However in an infinite dimensional framework several questions are still open. Our construction of generalized propagators for Radon measures (Section 3.2 below) can be compared with the strategy used by Miller and Rubinovich in Section 4.2.1 of [MR03] for finite-dimensional systems.

1.4 Main results

1.4.1 Upper bound for attainable sets of bilinear control systems

Our aim is to give upper bounds for the attainable set of a bilinear control system. The main result is the following.

Theorem 1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space, A be a maximal dissipative operator on $\mathcal{H}$ with domain $D(A)$, and B be an operator on $\mathcal{H}$ such that $B-c$ and $-B-c'$ generate contraction semigroups leaving $D(A)$ invariant for some real constants $c \geq 0$ and $c' \geq 0$. Assume that $A+uB$ is maximal dissipative with domain $D(A)$ for every u in $\mathbf{R}$ and that the map $t \in \mathbf{R} \mapsto e^{tB}Ae^{-tB} \in L(D(A), \mathcal{H})$, where $D(A)$ is endowed with the graph norm, is locally Lipschitz. Then, for every $T>0$ and for every ψ_0 in $\mathcal{H}$, there exists a unique continuous extension to $L^1([0,T],\mathbf{R})$ of the enpoint mapping $u \mapsto \Upsilon^u_T\psi_0 \in \mathcal{H}$ of (1.1), moreover the set*

$$\mathcal{A}tt_{L^1}(\psi_0) := \bigcup_{T\geq 0} \bigcup_{u\in L^1([0,T],\mathbf{R})} \{\Upsilon^u_{t,0}\psi_0 \mid t \in [0,T]\}$$

is contained in a countable union of compact subsets of $\mathcal{H}$.

Proof. See Section 3.3. □

When A and B are skew-adjoint, the orbits lie in the sphere of $\mathcal{H}$ of radius $\|\psi_0\|$, which is, of course, a meager set in $\mathcal{H}$. As a consequence of Theorem 1,

$$\bigcup_{\alpha\geq 0} \bigcup_{T\geq 0} \bigcup_{u\in L^1([0,T],\mathbf{R})} \{\alpha\Upsilon^u_{t,0}\psi_0 \mid t \in [0,T]\}$$

is a meager set in $\mathcal{H}$ and hence it has dense complement. Nonetheless, in the skew-adjoint case, the attainable set is a meager set (and hence has dense complement) in the sphere of radius $\|\psi_0\|$.

In the special case where the control operator B is bounded, using a different construction, we obtain the simplified statement below similar to the one of [BMS82], but dealing with L^1 controls.

Proposition 2. *Let $\mathcal{X}$ be an infinite dimensional Banach space, A generate a C^0 semigroup of bounded linear operators on $\mathcal{X}$, and B be a bounded linear operator on $\mathcal{X}$. Then for every $T>0$, there exists a unique continuous extension to $L^1([0,T],\mathbf{R})$ of the input-output mapping $u \mapsto \Upsilon^u_T \in L(\mathcal{H},\mathcal{H})$ of (1.1) and, for every ψ_0 in $\mathcal{H}$,*

$$\mathcal{A}tt_{L^1}(\psi_0) := \bigcup_{T\geq 0} \bigcup_{u\in L^1([0,T],\mathbf{R})} \{\Upsilon^u_t\psi_0 \mid t \in [0,T]\}$$

is contained in a countable union of compact subsets of $\mathcal{X}$ and, in particular, has dense complement.

Proof. See Section 5. □

These results set the open question by Ball, Marsden, and Slemrod in [BMS82, Remark 3.8]. The scheme of the proofs of Theorem 1 and Proposition 2 follows the structure of the proof of [BMS82, Theorem 3.6]. The lack of reflectiveness of L^1 leads us to consider Radon measures as controls, the weak-compactness of bounded sequences is ensured by Helly's Selection Theorem. The main difficulty is to define a continuous input-output mapping associated with (1.1) in such a way to guarantee compactness properties for the attainable sets.

Remark 2. Theorem 1 still holds true for Radon measures controls, as stated in Corollary 18 below. Here the result is presented in term of L^1 controls for the sake of readability, indeed the definition of propagator associated with a Radon measure requires preliminary notions presented in Section 3.2. The hypotheses of Theorem 1 are needed in order to prove continuity of the propagators after a particular change of variable (the *interaction framework* presented in Section 3). The key result in the proof of the continuity is an adaptation of a classical result by Kato [Kat53] (see Proposition 7).

1.4.2 Higher regularity

The Lipschitz assumption on the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ in Theorem 1 is crucial for our analysis when B is unbounded, however it may be hard to check in practice. For bilinear systems encountered in quantum physics, one can take advantage of the skew-adjointness of the operators to simplify the analysis. For instance, we have the following result.

Theorem 3. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space, k a positive number, A and B be two skew-adjoint operators such that:*

- (i) *A is invertible with bounded inverse from $D(A)$ to $\mathcal{H}$,*
- (ii) *for any $t \in \mathbf{R}$, $e^{tB} D(|A|^{k/2}) \subset D(|A|^{k/2})$,*
- (iii) *there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\|\cdot\|_{k/2}$,*
- (iv) *B is A-bounded with $\|B\|_A = 0$ (see (2.2) below for the precise definition).*

Then, for every $T > 0$, there exists a unique strongly continuous extension to $BV([0,T], \mathbf{R})$, endowed with the $\|\cdot\|_{L^1} + \mathrm{TV}(\cdot, ([0,T], \mathbf{R}))$-norm, of the end-point mapping $u \mapsto \Upsilon^u_T$ of (1.1). Moreover, for every ψ_0 in $D(|A|^{k/2})$, the set

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in BV([0,T],\mathbf{R})} \{\alpha \Upsilon^u_t \psi_0, t \in [0,T]\},$$

is contained in a countable union of compact subsets of $D(|A|^{k/2})$.

Proof. See Section 4.2. □

Remark 3. Theorem 3 is a reformulation of Theorem 1 in the smaller functional framework of conservative dynamics. The proof of Theorem 3 is a consequence of Corollary 26 below in the case of bounded variation controls. In Section 4.3 the result is then generalized to Radon measures controls. Corollary 26 also provides an extension of Theorem 3 from $D(|A|^{k/2})$ to $D(|A|^{k/2+1-\varepsilon})$ if ψ_0 is in $D(|A|^{k/2+1-\varepsilon})$, for $\varepsilon \in (0,1)$.

Remark 4. A simple checkable condition for a pair of skew-adjoint operators (A, B) to satisfy assumptions $(i) - (iii)$ in Theorem 3 is to be weakly coupled in the sense of [BCC13, Definition 1]. See Lemma 21 below.

Remark 5. Recall that there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ if and only if these operators are maximal dissipative in the functional space $D(|A|^{k/2})$. Assumption (iii) in Theorem 3 is, in some sense, an assumption on the commutator of A and B, see Section 4. This condition replaces the Lipschitz assumption on the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ of Theorem 1.

1.4.3 Applications to the bilinear Schrödinger equation

Here we consider the motion of a nonrelativistic quantum charged particle trapped in an infinite square potential well excited by an external electric field. That is the dynamics governed by a Schrödinger equation on the interval $(0,1)$ with a control potential $W : (0,1) \to \mathbf{R}$, which writes

$$\begin{cases} \mathrm{i}\dfrac{\partial \psi}{\partial t}(t,x) = -\dfrac{\partial^2 \psi}{\partial x^2}(t,x) - u(t) W(x) \psi(t,x), & x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0. \end{cases} \tag{1.5}$$

We denote by $H^s_{(0)}((0,1),\mathbf{C})$ the domain of $|A|^{s/2}$ where A is the Laplace–Dirichlet operator on $(0,1)$, and by φ_k, $k \in \mathbf{N}$ its (normalized) eigenvectors associated respectively to λ_k, $k \in \mathbf{N}$ its increasing sequence of eigenvalues (which are known to be simple). Let us recall the main result of [BL10].

Theorem (Theorem 1 in [BL10]). Let $T > 0$ and $W \in H^3((0,1),\mathbf{R})$ be such that there exists $c > 0$ verifying $\frac{c}{k^3} \leqslant |\langle W\varphi_1, \varphi_k \rangle|$, for all $k \in \mathbf{N}$. Then there exists $\delta > 0$ and a C^1 map $\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbf{R})$ where

$$\mathcal{V}_T := \{\psi_f \in H^3_{(0)}((0,1),\mathbf{C}) \mid \|\psi_f\| = 1, \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1.5) with initial condition $\psi(0) = \varphi_1$ and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

The above result applies for instance to $W : x \mapsto x^2$. Since the control potential is bounded, the input-output mapping $u \mapsto \Upsilon^u_T$ of (1.5) admits a unique continuous extension to $L^1([0,T],\mathbf{R})$. The techniques introduced in the present analysis provide the following estimates from above for the attainable set when using different classes of admissible controls.

Proposition 4. *Let $T > 0$ and $W : x \mapsto x^2$. Then:*

- *The attainable set from φ_1 with L^1 controls,*

$$\mathcal{A}tt_{L^1}(\varphi_1) = \bigcup_{T \geq 0} \bigcup_{u \in L^1([0,T],\mathbf{R})} \{\Upsilon^u_t \varphi_1 | 0 \leq t \leq T\},$$

satisfies $\mathcal{A}tt_{L^1}(\varphi_1) \subset \bigcap_{s<5/2} H^s_{(0)}((0,1),\mathbf{C})$.

- *The attainable set from φ_1 with bounded variation (BV) controls,*

$$\mathcal{A}tt_{BV}(\varphi_1) = \bigcup_{T \geq 0} \bigcup_{u \in BV((0,T],\mathbf{R})} \{\Upsilon^u_t \varphi_1 | 0 \leq t \leq T\},$$

is a H^s-dense subset of $\{\psi \in L^2((0,1),\mathbf{C}) \mid \|\psi\| = 1\} \cap H^s_{(0)}((0,1),\mathbf{C})$ for every $s < 9/2$.

Proof. See Section 6.2. ∎

1.5 Contents

In Section 2, we consider bilinear evolution equations (not necessarily conservative) from an abstract point of view and we define the solution for controls with bounded variations. We also prove the well-posedness within this framework and the continuity of the propagators with respect to the control parameters. In Section 3, we use a reparametrization, inspired by the widely used interaction framework, to extend the results of Section 2 to the case where the control is a Radon measure. This provides a proof of Theorem 1. Section 4 is devoted to the regularity analysis of the solution obtained so far when further assumptions are made on the control potential and to the proof of Theorem 3. Section 5 is dedicated to the case where B is bounded and to the proof of Proposition 2. Section 6 presents various examples. The appendices contain notations and technical tools useful for the rest of the analysis.

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2 Well-posedness and continuity for BV controls

In this section, we present global well-posedness results for a class of nonautonomous perturbations of a maximal dissipative linear Cauchy problem as well as a continuity criterion for a convergence problem.

2.1 Abstract framework: definitions and notations

We consider a general framework for bilinear dynamics in Hilbert spaces. Classical definitions and tools in this context can be found in [RS75, Section X.8], as well as the associated notes and problems. Notice that however we consider an opposite sign for the generators, thus, following [Phi59], we use the word *dissipative* instead of *accretive* (see also [RS75, Notes of Section X.8]). As we restrict our analysis to the Hilbert space framework, the notion of generators of contraction semigroups (linear maps with norm less than one) and maximal dissipative operators coincide (see [Phi59, Theorem 1.1.3]). The equivalence between these two notions is used in our analysis at many levels, in particular, for what concerns mild coupling in Section 4.

Let $\mathcal{H}$ be a Hilbert space (possibly infinite dimensional) with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Let A, B be two (possibly unbounded) dissipative operators on $\mathcal{H}$. We consider the formal bilinear control system

$$\frac{d}{dt}\psi(t) = A\psi(t) + u(t)B\psi(t), \tag{2.1}$$

where the scalar control u is to be chosen in a set of real functions.

In general, given an initial data $\psi(0) = \psi_0 \in \mathcal{H}$, the solution of system (2.1) may not be well-defined. Indeed, even the definition of $A + B$ is not obvious when A and B are unbounded. To this aim it is usually assumed that the operators A and B satisfy the following condition.

Definition 1. Let (A, B) be a couple of operators acting on $\mathcal{H}$. Then B is said *relatively bounded* with respect to A, or A-bounded, if $D(A) \subset D(B)$ and there exist $a, b > 0$ such that for every ψ in $D(A)$, $\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$.

It is well-known that if A is skew-adjoint and B skew-symmetric, from Kato–Rellich Theorem, (see for example [RS75, Theorem X.12]), if B is relatively bounded with respect to A, then for every real constant u such that $|u| < 1/a$ (with a from Definition 1), $A + uB$ is skew-adjoint with domain $D(A)$ and generates a group of unitary operators. System (2.1) is then well-posed for every initial condition. From [RS75, Corollary to Theorem X.50], $A + uB$ is maximal dissipative with domain $D(A)$ and generates a contraction semigroup when A is maximal dissipative, B is dissipative, B is A-bounded and $0 \leq u < 1/a$ (again a is from Definition 1).

In most of the examples presented in Section 6 below, we consider the skew-adjoint case and a arbitrary small, so that we can define the solutions of (2.1) for every piecewise constant control u with real values.

In the general case, we will refer to the following assumptions.

Assumption 1. (A, B, K) is a triple where A is a maximal dissipative operator on $\mathcal{H}$, B is an operator on $\mathcal{H}$ with $D(A) \subset D(B)$, and K a real interval containing 0, such that for any $u \in K$, $A + uB$ is a maximal dissipative operator on $\mathcal{H}$ with domain $D(A)$.

Assumption 1 implies that the operator B is A-bounded from $D(A)$ to $\mathcal{H}$ and allows us to define

$$\|B\|_A := \inf_{\lambda > 0} \|B(\lambda - A)^{-1}\|. \tag{2.2}$$

The number $\|B\|_A$ is the lower bound of all possible constants a in Definition 1 and in principle it can be zero. We also have,

$$\|B\|_A = \liminf_{\lambda \to +\infty} \|B(\lambda - A)^{-1}\|. \tag{2.3}$$

We consider also the following assumption in order to extend the definition of propagator to the case of Radon measures controls (see Section 3.2).

Assumption 2. (A, B, K) is a triple where A is a maximal dissipative operator on $\mathcal{H}$, K a real interval containing 0, and

(A2.1) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $\mathcal{H}$ leaving $D(A)$ invariant,

(A2.2) for every $u \in \mathcal{R}([0, T])$, with $u((0, t]) \in K$ for any $t \in [0, T]$,

$$t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0,t])B} A e^{-u((0,t])B},$$

is a family of maximal dissipative operators with common domain $D(A)$ such that :

- $\sup_{t \in [0,T]} \left\|(1 - \mathcal{A}(t))^{-1}\right\|_{L(\mathcal{H}, D(A))} < +\infty$,
- $\mathcal{A}$ has finite total variation from $[0, T]$ to $L(D(A), \mathcal{H})$.

A basic example of operators satisfying Assumption 2 is given by $A = -\partial_x$ and $B = \mathrm{i}W(x)$, the operator of multiplication by a smooth bounded potential $\mathrm{i}W$ (W real-valued), acting on $L^2(\mathbf{R})$ (the set K being the real line $\mathbf{R}$). Then $\mathcal{A}(t) = -\partial_x + u((0, t])(\partial_x W)$.

Remark 6. From Assumption 2, B et $-B$, with same domains, are generators of continuous semi-groups. We can prove $e^{-tB} = (e^{tB})^{-1}$, for any real t, and thus B generates a continuous group.

The triple (A, B, K) satisfies Assumption 2 for any interval K containing 0 if the pair (A, B) satisfies the following one.

Assumption 3. (A, B) is a pair such that

(A3.1) A is a maximal dissipative operator on $\mathcal{H}$ with domain $D(A)$,

(A3.2) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $\mathcal{H}$ leaving $D(A)$ invariant,

(A3.3) the map $t \in \mathbf{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ is locally Lipschitz.

Remark 7. Assumption (A3.3) is a strong assumption on the regularity of B with respect to the scale of A. Indeed it implies that B is the generator of a strongly continuous semigroup on $D(A)$ since the semigroups generated by B or $-B$ are continuous on $\mathcal{H}$ from Assumption (A3.2) and

$$\begin{aligned} \left\|Ae^{-tB}\psi - A\psi\right\| &\leq e^{c't}\left\|e^{tB}Ae^{-tB}\psi - e^{tB}A\psi\right\| \\ &\leq e^{c't}\left\|e^{tB}Ae^{-tB} - A\right\|_{L(D(A),\mathcal{H})}\|\psi\|_{D(A)} + \left\|e^{tB}A\psi - A\psi\right\|, \end{aligned}$$

for $t > 0$ and $\psi \in D(A)$, which provides the continuity on $D(A)$. In Section 4 below, we consider higher regularity assumptions in the skew-adjoint case and operators on $D(|A|^k)$ with $k > 1$.

2.2 Propagators

Since the problem (2.1) is nonautonomous, the notion of semigroup is replaced by the following

Definition 2 (Propagator on a Hilbert space)**.** A family $(s, t) \in \Delta_I \mapsto X(s, t)$ of linear contractions on a Hilbert space $\mathcal{H}$, strongly continuous in t and s and such that

(*i*) $X(t, s) = X(t, r)X(r, s)$, for any $s < r < t$,

(*ii*) $X(t, t) = I_{\mathcal{H}}$,

is called a *contraction propagator* on $\mathcal{H}$.

Remark 8. In Section 3 below, we introduce a notion of generalized propagators, see Definition 4, with relaxed assumptions on the continuity of $(s, t) \mapsto X(s, t)$ in order to extend it to the framework of Radon measure controls.

Following [Kat53] in the construction of propagators, we introduce the following

Assumption 4. Let $\mathcal{D}$ be a dense subset of $\mathcal{H}$

(A4.1) $A(t)$ is a maximal dissipative operator on $\mathcal{H}$ with domain $\mathcal{D}$ for every $t \in I$,

(A4.2) $t \mapsto A(t)$ has bounded variation from I to $L(\mathcal{D}, \mathcal{H})$, where $\mathcal{D}$ is endowed with the graph topology associated with $A(a)$ for some $a \in I$,

(A4.3) $M := \sup_{t \in I} \left\|(1 - A(t))^{-1}\right\|_{L(\mathcal{H}, \mathcal{D})} < \infty$.

In the following, Assumption 4 will apply mainly to the family of operators $A(t) = A + u(t)B$ or $A(t) = e^{-u((0,t])B} A e^{u((0,t])B}$.

Remark 9. In Assumption (A4.2), the bounded variation of $t \mapsto A(t)$ ensures that any choice of $a \in I$ will be equivalent.

Remark 10. We do not assume $t \mapsto A(t)$ to be continuous. However, as a consequence of Assumption (A4.2) (see [Edw57, Theorem 3]) it admits right and left limit in $L(\mathcal{D}, \mathcal{H})$, denoted by $A(t-0) := \lim_{\varepsilon \to 0^+} A(t-\varepsilon)$ and $A(t+0) := \lim_{\varepsilon \to 0^+} A(t+\varepsilon)$, for all $t \in I$, and $A(t-0) = A(t+0)$ for all $t \in I$ except, at most, a countable set.

The core of our analysis is the following result due to Kato (see [Kat53, Theorem 2 and Theorem 3]) providing sufficient conditions for the well-posedness of system (2.1).

Theorem 5. *If $t \in I \mapsto A(t)$ satisfies Assumption 4, then there exists a unique contraction propagator $X : \Delta_I \to L(\mathcal{H})$ such that if $\psi_0 \in \mathcal{D}$ then $X(t,s)\psi_0 \in \mathcal{D}$ and is strongly right differentiable in t with derivative $A(t+0)X(t,s)\psi_0$.*

Moreover, with M from Assumption (A4.3),

$$\|A(t)X(t,s)\psi_0\| \leq M e^{M \mathrm{TV}(A,(I,L(\mathcal{D},\mathcal{H})))} \|A(s)\psi_0\|, \qquad \text{for } (t,s) \in \Delta_I \text{ and } \psi_0 \in \mathcal{D},$$

and $X(t,s)\psi_0$ is left differentiable in s with derivative $-A(s-0)\psi_0$ when $t = s$.

In the case in which $t \mapsto A(t)$ is continuous and skew-adjoint, if $\psi_0 \in \mathcal{D}$ then $t \in (s, +\infty) \mapsto X(t,s)\psi_0$ is strongly continuously differentiable in $\mathcal{H}$ with derivative $A(t)X(t,s)\psi_0$.

Proof. The statement of this theorem is obtained by collecting statements of [Kat53]. The point not clearly stated in [Kat53] is the existence of $C > 0$ such that

$$\|A(t)X(t,s)\psi_0\| \leq C\|A(s)\psi_0\|,$$

for $(t,s) \in \Delta_I$ and for any $\psi_0 \in \mathcal{D}$. This is in [Kat53, §3.10] with $C = M \exp(MN)$ and

$$M = \sup_{t \in I} \left\| (1 - A(t))^{-1} \right\|_{L(\mathcal{H},\mathcal{D})} \qquad \text{and} \qquad N = \mathrm{TV}(A, (I, L(\mathcal{D}, \mathcal{H}))). \qquad \square$$

We call $t \mapsto X(t,s)\phi_0$ a "mild" solution in $\mathcal{H}$ of

$$\begin{cases} \frac{d}{dt}\phi(t) = A(t)\phi(t), \\ \phi(s) = \phi_0, \end{cases} \tag{2.4}$$

even if, in general, it is not differentiable.

Remark 11. If (A, B, K) satisfies Assumption 2, the operator $t \in [0,T] \mapsto \mathcal{A}(t) := e^{u((0,t])B} A e^{-u((0,t])B}$ defined in Assumption (A2.2) satisfies Assumption 4 for any Radon measure u on $(0,T)$ with $u((0,t]) \in K$ for any $t \in (0,T]$. If (A,B) satisfies Assumption 3 then $(A, B, \mathbf{R})$ satisfies Assumption 2.

The fact that Assumption 1 is stronger, in some sense, than Assumption 4 is the content of the following lemma.

Lemma 6. *If (A, B, K) satisfies Assumption 1 and $u : [0,T] \mapsto K$ has bounded total variation such that $\overline{u([0,T])} \subset K$ then $A(t) := A + u(t)B$ satisfies Assumption 4 with $I = [0,T]$.*

Proof. The only point to verify is Assumption (A4.3). First the set $C := \overline{u([0,T])}$ is a bounded closed subset of K and thus is a compact of K. Then the map

$$u \mapsto (1 - A)(1 - A - uB)^{-1},$$

is continuous from K to $L(\mathcal{H})$. Indeed

$$\begin{aligned} (1 - A)(1 - A - uB)^{-1} - &(1 - A)(1 - A - vB)^{-1} \\ &= (1 - A)\left((1 - A - uB)^{-1} - (1 - A - vB)^{-1}\right) \\ &= (v - u)(1 - A)\left((1 - A - uB)^{-1} B (1 - A - vB)^{-1}\right) \\ &= (v - u)(1 - A)\left((1 - A - uB)^{-1} B (1 - A)^{-1}(1 - A)(1 - A - vB)^{-1}\right) \end{aligned}$$

so that

$$
\begin{aligned}
(1-&A)(1-A-uB)^{-1}-(1-A)(1-A-vB)^{-1} \\
&-(v-u)(1-A)(1-A-uB)^{-1}B(1-A)^{-1}\left((1-A)(1-A-vB)^{-1}-(1-A)(1-A-uB)^{-1}\right) \\
&=(v-u)(1-A)\left((1-A-uB)^{-1}B(1-A)^{-1}(1-A)(1-A-uB)^{-1}\right).
\end{aligned}
$$

Define

$$
L(u)=\|(1-A)(1-A-uB)^{-1}\|_{L(\mathcal{H})} \quad \text{and} \quad b=\|B(1-A)^{-1}\|,
$$

so that

$$
(1-|v-u|bL(u))\|(1-A)(1-A-uB)^{-1}-(1-A)(1-A-vB)^{-1}\| \leq |v-u|L(u)^2 b, \tag{2.5}
$$

which provides the desired continuity at u. Then as $|u(t)-u(0)| \leq \|u\|_{BV(I)}$ for any $t \in I$, $u(t)$ is in C a compact subset of K for all $t \in I$ thus the closure of its image is compact and

$$
t \in I \mapsto \|(1-A-u(t)B)^{-1}\|_{L(\mathcal{H},\mathcal{D})}
$$

is bounded. ☐

2.3 Continuity

In this section we focus on the continuity of the propagators with respect to the control u. The main tool, Proposition 7 below, is a consequence of the work [Kat53] by Kato.

Definition 3. Let $(A_n)_n$ be a family of generators of contraction semigroups and A a generator of a contraction semigroup. The family $(A_n)_n$ tends to A in the *strong resolvent sense* if

$$
(\lambda-A_n)^{-1}\phi \to (\lambda-A)^{-1}\phi \quad \text{as } n \to \infty,
$$

for every ϕ in $\mathcal{H}$ and for some $\lambda > 0$ (and hence all $\lambda > 0$, see [RS72, Section VIII.7]).

Proposition 7. *Let $(A_n)_{n\in\mathbf{N}}$ and A satisfy Assumption 4. Let $(\mathcal{D}_n)_{n\in\mathbf{N}}$ and $\mathcal{D}$ be their respective domains (for any $t \in I$). Let X_n (respectively X) be the contraction propagator associated with A_n (respectively A).*

Assume that:

(*i*) $\sup_{n\in N}\sup_{t\in I}\|(1-A_n(t))^{-1}\|_{L(\mathcal{H},\mathcal{D}_n)} < +\infty$,

(*ii*) *$A_n(\tau)$ converges to $A(\tau)$ in the strong resolvent sense for almost every $\tau \in I$ as $n \to \infty$,*

(*iii*) $\sup_{n\in\mathbf{N}} \mathrm{TV}(A_n,(I,L(\mathcal{D}_n,\mathcal{H}))) < +\infty$,

(*iv*) *For every $\phi \in \mathcal{H}$, $\delta > 0$, $n \in \mathbf{N}$ there exists $\psi^n \in \mathcal{D}_n$ with $\|\phi-\psi^n\| < \delta$ such that $\sup_{n\in\mathbf{N}}\|A_n(a)\psi^n\| < +\infty$ for some $a \in I$.*

Then $X_n(t,s)$ tends strongly to $X(t,s)$ locally uniformly in $s,t \in \Delta_I$.

Proof. Using [Kat53, §3.8] it is sufficient to prove the statement for piecewise constant operator-valued functions (i.e. replacing X_n and X by any of their Riemann products) as follows: Let $\Delta := \{s=t_0<t_1<\ldots<t_n=t\}$ be a partition of the interval (t,s) and $X_n(\Delta)$ be the propagator associated with $\sum_{j=1}^{n} A_n(t_{j-1})\chi_{[t_{j-1},t_j)}$. Then, from [Kat53, Equation (3.16)], for every n,

$$
\|\left(X_n(t,s;\Delta)-X_n(t,s)\right)\phi\| \leq \overline{M}e^{\overline{M}\,\overline{N}}\overline{N}|\Delta|\|A_n(a)\phi\|, \text{ for every } \phi \in \mathcal{D}_n
$$

where

$$\overline{M} = \max\{\sup_{t\in I}\sup_{n\in\mathbf{N}} \left\|(1-A_n(t))^{-1}\right\|_{L(\mathcal{H},\mathcal{D}_n)}, \sup_{t\in I}\left\|(1-A(t))^{-1}\right\|_{L(\mathcal{H},\mathcal{D})}\},$$
$$\overline{N} = \max\{\sup_{n\in\mathbf{N}} \mathrm{TV}(A_n,(I,L(\mathcal{D}_n,\mathcal{H}))), \mathrm{TV}(A,(I,L(\mathcal{D},\mathcal{H})))\},$$

and $|\Delta| = \sup_{1\leq j\leq n}|t_j - t_{j-1}|$. Similarly we define $X(\Delta)$ as the propagator associated with

$$\sum_{j=1}^{n} A(t_{j-1})\chi_{[t_{j-1},t_j)}.$$

We have

$$\|\left(X(t,s;\Delta)-X(t,s)\right)\phi\| \leq \overline{M}e^{\overline{M}\,\overline{N}}\overline{N}|\Delta|\|A(a)\phi\|, \text{ for every } \phi\in\mathcal{D}.$$

Following the proof of [RS75, Theorem X.47a (Hille–Yosida)] (see also Proposition 19 below), we have that

$$\left\|e^{tA_n(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi\right\| \leq t\left\|A_n(\tau)\phi - A_n^\lambda(\tau)\phi\right\|, \quad \text{for every } \phi\in\mathcal{D}_n,$$

with $A_n^\lambda(\tau) := \lambda(\lambda - A_n(\tau))^{-1}A_n(\tau)$, for $\lambda > 0$. This estimates can be obtained by integrating for $s\in[0,t]$ the derivative with respect to s of

$$e^{sA_n(\tau)}e^{(t-s)A_n^\lambda(\tau)}\phi,$$

using the fact that $A_n(\tau)$ and $A_n^\lambda(\tau)$ commute, the triangle inequality, and the fact that the $e^{sA_n(\tau)}$ and $e^{(t-s)A_n^\lambda(\tau)}$ are contractions (see, for instance, [RS75, Proof of Theorem X.47a (Hille–Yosida)].

We also have

$$\left\|e^{tA(\tau)}\phi - e^{tA^\lambda(\tau)}\phi\right\| \leq t\left\|A(\tau)\phi - A^\lambda(\tau)\phi\right\|, \quad \text{for every } \phi\in\mathcal{D},$$

with $A^\lambda(\tau) := \lambda(\lambda - A(\tau))^{-1}A(\tau)$.

Since A_n are generators of contraction semigroups, then $\|\lambda(\lambda - A_n(\tau))^{-1}\| \leq 1$ for every $\lambda > 0$, in particular it is uniformly bounded in n and τ.

By assumption (iv) for every $\phi\in\mathcal{H}$ and $\delta > 0$ there exist $\psi\in\mathcal{D}$ and $\psi^n\in\mathcal{D}_n$ such that

$$\|\phi-\psi\|\leq\delta \quad \text{and} \quad \|\phi-\psi^n\|\leq\delta,$$

and $\sup_{n\in\mathbf{N}}\|A_n(a)\psi^n\| < +\infty$ for $a\in I$. We deduce that $\lambda(\lambda - A_n(\tau))^{-1}\psi^n$ tends to ψ^n as $\lambda\to\infty$ uniformly in n and τ. Similarly $A^\lambda(\tau)\psi$ tends strongly to $A(\tau)\psi$ uniformly in τ as $\lambda\to\infty$. So that

$$\begin{aligned}\left\|e^{tA(\tau)}\phi - e^{tA_n(\tau)}\phi\right\| &\leq 2\delta + \left\|e^{tA(\tau)}\psi - e^{tA^\lambda(\tau)}\psi\right\| + \left\|e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi\right\| + \left\|e^{tA_n^\lambda(\tau)}\psi^n - e^{tA_n(\tau)}\psi^n\right\| \\ &\leq 2\delta + t\left\|A(\tau)\psi - A^\lambda(\tau)\psi\right\| + \left\|e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi\right\| + t\left\|A_n(\tau)\psi^n - A_n^\lambda(\tau)\psi^n\right\|.\end{aligned}$$

It is sufficient to show convergence of $\left\|e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi\right\|$ as $n\to\infty$ in order to conclude the proof. Since $e^{tA_n^\lambda(\tau)} = e^{-\lambda t}e^{t\lambda^2(\lambda-A_n(\tau))^{-1}}$ and $e^{tA^\lambda(\tau)} = e^{-\lambda t}e^{t\lambda^2(\lambda-A(\tau))^{-1}}$ (see [RS75, Theorem X.47a (Hille-Yosida)]), we have that

$$\begin{aligned}\left\|e^{tA^\lambda(\tau)}\phi - e^{tA_n^\lambda(\tau)}\phi\right\| &= \left\|e^{-\lambda t}e^{t\lambda^2(\lambda-A_n(\tau))^{-1}}\phi - e^{-\lambda t}e^{t\lambda^2(\lambda-A(\tau))^{-1}}\phi\right\| \\ &= e^{-\lambda t}\left\|e^{t\lambda^2(\lambda-A_n(\tau))^{-1}}\phi - e^{t\lambda^2(\lambda-A(\tau))^{-1}}\phi\right\|.\end{aligned}$$

Now, we have that $\|(\lambda - A_n(\tau))^{-1}\| \leq \frac{1}{\lambda}$ (see Proposition 19 below for $\omega = 0$) and hence $\|e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\| \leq e^{\lambda t}$. Duhamel's identity then writes, for $0 \leq t \leq T$,

$$\begin{aligned}
&\left\| e^{t\lambda^2(\lambda - A_n(\tau))^{-1}}\phi - e^{t\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\| \\
&= \left\| \int_0^t \lambda^2 e^{(t-s)\lambda^2(\lambda - A_n(\tau))^{-1}} \left\{ (\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1} \right\} e^{s\lambda^2(\lambda - A(\tau))^{-1}}\phi \, \mathrm{d}s \right\| \qquad (2.6) \\
&\leq \lambda^2 e^{T\lambda} \int_0^T \left\| \left\{ (\lambda - A_n(\tau))^{-1} - (\lambda - A(\tau))^{-1} \right\} e^{s\lambda^2(\lambda - A(\tau))^{-1}}\phi \right\| \mathrm{d}s.
\end{aligned}$$

The result follows from Lebesgue Dominated Convergence Theorem, using the convergence of $A_n(\tau)$ to $A(\tau)$ in the strong resolvent sense for almost every $\tau \in I$ as n tends to infinity. $\square$

Lemma 8. *Let $(A_n)_{n \in \mathbf{N}}$ and A satisfy Assumption 4 with a common domain $\mathcal{D}$ (for any $t \in I$ and any $n \in \mathbf{N}$). Let X_n, respectively X, be the contraction propagator associated with A_n, respectively A.*

Then the assumptions of Proposition 7 are verified whenever:

$(i)'$ $\sup_{n \in \mathbf{N}} \sup_{t \in I} \|(1 - A_n(t))^{-1}\|_{L(\mathcal{H}, \mathcal{D})} < +\infty$,

$(ii)'$ *$A_n(\tau)$ converges to $A(\tau)$ in the strong sense in $\mathcal{D}$ for almost every $\tau \in I$ as $n \to \infty$,*

$(iii)'$ $\sup_{n \in \mathbf{N}} \mathrm{TV}(A_n, (I, L(\mathcal{D}, \mathcal{H}))) < +\infty$.

Proof. Assumptions (i) and (iii) of Proposition 7 coincide, respectively, with assumptions $(i)'$ and $(iii)'$.

We have for any ϕ in $\mathcal{H}$

$$(1 - A_n(t))^{-1}\phi - (1 - A(t))^{-1}\phi = (1 - A_n(t))^{-1}(A(t) - A_n(t))(1 - A(t))^{-1}\phi$$

and hence

$$\left\|(1 - A_n(t))^{-1}\phi - (1 - A(t))^{-1}\phi\right\| \leq \left\|(A(t) - A_n(t))(1 - A(t))^{-1}\phi\right\|.$$

Since $(1 - A(t))^{-1}\phi \in \mathcal{D}$ by assumption $(ii)'$ we conclude that (ii) of Proposition 7 is verified as well.

As $\mathcal{D}$ is dense for every $\phi \in \mathcal{H}$, $\delta > 0$, there exists $\psi \in \mathcal{D}$ with $\|\phi - \psi\| < \delta$. Since for any $a \in I$ $A_n(a)$ converges strongly to $A(a)$ in $\mathcal{D}$, $\sup_{n \in \mathbf{N}} \|A_n(a)\psi\| < +\infty$. This is assumption (iv) of Proposition 7 $\square$

Corollary 9. *Let (A, B, K) satisfy Assumption 1. Let $(u_n)_{n \in N}$ be a sequence in $BV(I, K)$ converging to $u \in BV(I, K)$. Let $A_n(t) = A + u_n(t)B$, $A(t) = A + u(t)B$ and let X_n, respectively X, be the contraction propagators associated with A_n, respectively A. If $\overline{\cup_{n \in \mathbf{N}} u_n([0, T])} \subset K$, then $X_n(t, s)$ tends strongly to $X(t, s)$ locally uniformly in $(s, t) \in \Delta_I$.*

Proof. The proof consists in verifying that the hypotheses of Proposition 7 are satisfied. To this aim, we just have to check items $(i)'$, $(ii)'$ and $(iii)'$ of Lemma 8.

Assumption $(i)'$: the mapping $L : s \in K \mapsto \|(1 - A)(1 - A - sB)^{-1}\|$ has been defined in the proof of Lemma 6 where it is shown to be continuous. By hypothesis, there exists a compact set $K_1 \subset K$ such that for every n in $\mathbf{N}$ and every t in $[0, T]$, $u_n(t) \in K_1$. Hence, $\sup_{n \in \mathbf{N}} \sup_{y \in [0,T]} \|(1 - A)(1 - A - u_n(t)B)^{-1}\| \leq C(K_1) < +\infty$ which proves point $(i)'$.

Assumption $(ii)'$ follows from the assumption that $(u_n)_{n \in \mathbf{N}}$ converges to u.

Assumption $(iii)'$: for every n in $\mathbf{N}$,

$$\mathrm{TV}(A_n, (I, L(\mathcal{D}, \mathcal{H}))) = \mathrm{TV}(u_n B, (I, L(\mathcal{D}, \mathcal{H}))) = \|B\|_{L(\mathcal{D}, \mathcal{H})} \mathrm{TV}(u_n, (I, \mathbf{R})).$$

This last quantity is bounded as n tends to infinity since $(u_n)_{n \in \mathbf{N}}$ converges to u. $\square$

Corollary 10. *Assume that* (A, B, K) *satisfy Assumption 1. Let* $\psi_0 \in \mathcal{H}$. *Then*

$$\{\Upsilon_t^u(\psi_0) \mid u \in BV([0,\infty), K), t \geq 0\}$$

is contained in a countable union of compact subsets of $\mathcal{H}$.

Proof. We follow the principle presented in Section 1.1.2. We first introduce a nondecreasing sequence $(K_i)_{i\in\mathbf{N}}$ of compact subsets of K such that $K = \cup_{i\in\mathbf{N}} K_i$, and the subsets

$$\mathcal{Z}_{i,j,n} = \{u \in BV([0,\infty), K_i), \mathrm{TV}(u, ([0,n], K_i)) \leq j\}$$

of the set of functions with bounded variations. By Helly's selection Theorem, $\mathcal{Z}_{i,j,n}$ is sequentially compact. By Corollary 9, the set $\{\Upsilon_t^u(\psi_0) \mid u \in \mathcal{Z}_{i,j,n}\}$ is compact in $\mathcal{H}$ for every (n, i, j) in $\mathbf{N}^3$. Hence

$$\begin{aligned}&\{\Upsilon_t^u(\psi_0) \mid u \in BV([0,\infty), K), t \geq 0\} \subset \\ &\qquad \cup_{n\in\mathbf{N}} \cup_{i\in\mathbf{N}} \cup_{j\in\mathbf{N}} \{\Upsilon_t^u(\psi_0) \mid u \in \mathcal{Z}_{i,j,n}, 0 \leq t \leq n\}\end{aligned}$$

is contained is a countable union of compact sets of $\mathcal{H}$. □

The notion of convergence for a sequence of Radon measures is detailed in Appendix A.

Corollary 11. *Let* (A, B, K) *satisfy Assumption 2. Let* $I = [0, T]$ *for some* $T > 0$. *Let* $(v_n)_{n\in N}$ *be a sequence in* $\mathcal{R}(I)$ *converging to* $v \in \mathcal{R}(I)$. *Assume that* $v_n((0,t]) \in K$ *and* $v((0,t]) \in K$ *for every* $t \in (0, T]$ *and* $n \in \mathbf{N}$. *Let* $\mathcal{A}_n(t) = e^{-v_n((0,t])B} A e^{v_n((0,t])B}$ *and* $\mathcal{A}(t) = e^{-v((0,t])B} A e^{v((0,t])B}$ *and let* X_n, *respectively* X, *be the contraction propagators associated with* $\mathcal{A}_n$, *respectively* $\mathcal{A}$. *If* $\sup_{n\in\mathbf{N}} \mathrm{TV}(\mathcal{A}_n, (I, L(D(A), \mathcal{H}))) < +\infty$, *then* $X_n(t, s)$ *tends strongly to* $X(t, s)$ *locally uniformly in* $(s, t) \in \Delta_I$.

Proof. The proof consists in checking that the assumptions of Proposition 7 are fulfilled. Here $\mathcal{D} = D(A)$.

(*i*) We have $\sup_{n\in N} \sup_{t\in I} \|(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H},\mathcal{D})} < \infty$. Indeed

$$\begin{aligned}
&\|(1 - A)(1 - \mathcal{A}_n(t))^{-1}\|_{L(\mathcal{H})} \\
&= \|(1 - A)e^{v_n((0,t])B}(1 - A)^{-1}e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})}\|e^{-v_n((0,t])B}(1 - A)e^{v_n((0,t])B}(1 - A)^{-1}\|_{L(\mathcal{H})}\|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\
&= \|e^{v_n((0,t])B}\|_{L(\mathcal{H})}\|(1 - \mathcal{A}_n(t))(1 - A)^{-1}\|_{L(\mathcal{H})}\|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} \left(\|\mathcal{A}_n(t) - \mathcal{A}_n(0)\|_{L(\mathcal{D},\mathcal{H})} + \|1 - A\|_{L(\mathcal{D},\mathcal{H})}\right) \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})} \\
&\leq \|e^{v_n((0,t])B}\|_{L(\mathcal{H})} \left(\mathrm{TV}(\mathcal{A}_n, (I, L(\mathcal{D}, \mathcal{H}))) + 1\right) \|e^{-v_n((0,t])B}\|_{L(\mathcal{H})}.
\end{aligned} \tag{2.7}$$

Notice that since $(v_n)_{n\in N}$ converges to v then by definition $v_n((0,t])$ is uniformly bounded in $n \in \mathbf{N}$ and $t \in [0, T]$. Then from Assumption (A2.1), there exists $\omega \in \mathbf{R}$ such that

$$\|e^{vB}\|_{L(\mathcal{H})} \leq e^{\omega|v|}, \quad \text{for every } v \in \mathbf{R}, \tag{2.8}$$

which provides the desired boundedness.

(*ii*) The sequence $\mathcal{A}_n(t)$ tends to $\mathcal{A}(t)$ in the strong resolvent sense for all $t \in [0, T]$ as $n \to \infty$. Indeed from

$$(1 - \mathcal{A}_n(t))^{-1} - (1 - \mathcal{A}(t))^{-1} = e^{-v_n((0,t])B}(1 - A)^{-1}e^{v_n((0,t])B} - e^{-v((0,t])B}(1 - A)^{-1}e^{-v((0,t])B}$$

we have

$$(1-\mathcal{A}_n(t))^{-1}-(1-\mathcal{A}(t))^{-1}=(e^{-v_n((0,t])B}-e^{-v((0,t])B})(1-A)^{-1}e^{v_n((0,t])B}$$
$$+e^{-v((0,t])B}(1-A)^{-1}(e^{v_n((0,t])B}-e^{v((0,t])B})$$

then using (2.8) the boundedness of the sequence (v_n) and the strong continuity of $t\in\mathbf{R}\mapsto e^{tB}$, we conclude the strong resolvent convergence.

(*iii*) By Assumption (A2.2) we have $\sup_{n\in\mathbf{N}}\mathrm{TV}(\mathcal{A}_n,(I,L(D(A),\mathcal{H})))<+\infty$.

(*iv*) Assumption (*iv*) of Proposition 7 follows from $\mathcal{A}_n(0)=A$ and the fact that the domain $\mathcal{D}$ of A is dense in $\mathcal{H}$. □

Remark 12. The last assumption of Corollary 11, namely $\sup_{n\in\mathbf{N}}\mathrm{TV}(\mathcal{A}_n,(I,L(\mathcal{D},\mathcal{H})))<+\infty$ for $\mathcal{A}_n(t)=e^{-u_n((0,t])B}Ae^{u_n((0,t])B}$, is a consequence of Assumption (A3.3) since this provides the existence of a real constant $L_I(A,B)$ such that for every $s,t\in I$,

$$\|e^{-tB}Ae^{tB}-e^{-sB}Ae^{sB}\|_{L(\mathcal{D},\mathcal{H})}\leq L_I(A,B)|t-s|. \tag{2.9}$$

Notice also that with $s=0$ inequality (2.9) reads

$$\|e^{-tB}Ae^{tB}\|_{L(\mathcal{D},\mathcal{H})}\leq L_I(A,B)|t|+1 \tag{2.10}$$

as $\|A\|_{L(\mathcal{D},\mathcal{H})}\leq 1$.

3 Interaction framework

In this section we consider the framework of assumptions 2 or 3. We show that these assumptions lead to a notion of weak solution for (2.1) when the control is integrable and we provide the proofs of Theorem 1 and Proposition 2 in the general Radon measure case.

3.1 Heuristic

A classical method to deal with time-depending Hamiltonians, as it is the case of bilinear dynamics of the form (1.1), is to considered solution in the mild sense

$$x(t)=e^{tA}x_0+\int_0^t e^{(t-s)A}u(s)Bx(s)\mathrm{d}s,$$

using the well-known *interaction picture*, consisting in a change of variable $y(t)=e^{-tA}x(t)$ which gives (formally) $y'(t)=u(t)e^{-tA}Be^{tA}y(t)$. The underlying idea is that the operator uB is “small” with respect to A and, hence, y is expected to have slow variations. However, in our framework, since we consider control u in the set of Radon measures, we need a different approach. Indeed when u has atoms, the effect of A on the dynamics is negligible compared to uB. So we consider the change of variables $y(t)=e^{-\int u(t)B}x(t)$, and hence (formally) $y'=e^{-\int uB}Ae^{\int uB}y$, to finally obtain mild solutions of (1.1) in a generalized mild sense

$$x(t)=e^{\int_0^t uB}x_0+\int_0^t e^{\int_s^t uB}Ax(s)\mathrm{d}s. \tag{3.1}$$

This heuristic is purely formal at this point and the aim of the rest of this section is to formalize it rigorously. In the formal expression (3.1), one of the difficulties lies in the interpretation of the term $\int_0^t u(s)\mathrm{d}s$ when u is a Radon measure and $u(\{t\})\neq 0$: should the limits of $[0,t]$ be included in the integral? This is also the reason for the introduction of a non-standard notion of sequential convergence on the set of Radon measures, needed in order to ensure the continuity of the endpoint mapping.

3.2 Generalized propagators

In this section, we explain the link between Assumption 1 and Assumption 2 and thus emphasize the fact that (2.1) admits solutions associated with a Radon measure u.

We use the following result of approximation of Radon measures by piecewise constant functions.

Lemma 12. *For every $u \in \mathcal{R}([0,T])$ there exists a sequence $(u_n)_n$ of piecewise constant functions such that $\int_0^t u_n$ tends to $u((0,t])$ and $\int_0^t |u_n|$ tends to $|u|((0,t])$ for all t in $[0,T]$ as n tends to infinity with $\int_0^T |u_n| \leq |u|((0,T])$ for every n. If u is positive, the sequence $(u_n)_n$ can be chosen such that $t \mapsto \int_0^t u_n(\tau)\mathrm{d}\tau$ is nondecreasing for every n. If $t \mapsto u((0,t])$ is M-Lipschitz continuous on $[0,T]$ then $(u_n)_n$ can be chosen such that $|u_n| \leq M$.*

Proof. It is not restrictive to prove the statement for nonnegative Radon measures since by Hahn–Jordan decomposition, any Radon measure u is the difference of two nonnegative Radon measures with disjoint supports.

Let us assume u nonnegative. Then $U : t \in (0,T] \mapsto u((0,t])$ is an nondecreasing function (with bounded variation). Except on an at most countable set, U is continuous. So U is the sum of an nondecreasing step function, possibly with an infinite number of steps, and a nondecreasing continuous function. Both can be pointwise approximated by nondecreasing sequences of nondecreasing continuous piecewise affine functions.

The last statement follows by considering approximation of Lipschitz continuous functions by continuous piecewise affine ones. $\square$

Remark 13. Lemma 12 explains the reason for the choice of the notion of convergence for sequences on Radon measure, given in Appendix A, in stead of the maybe more natural total variation topology. Indeed, for a positive $u \in \mathcal{R}([0,T])$ the sequence $(u_n)_n$ of piecewise constant functions such that $\int_0^t u_n$ tends to $u((0,t])$ pointwise is, in our construction, a nondecreasing sequence. Since each $t \mapsto \int_0^t u_n$ is continuous, if $t \mapsto u((0,t])$ is not continuous then a result such as Lemma 12 in the total variation topology is excluded. Indeed the convergence in total variation of sequences of measures implies the uniform convergence of the corresponding sequence of cumulative functions.

Definition 4. Let (A, B, K) satisfy Assumption 2. Let $u \in \mathcal{R}([0,T])$. For any $v \in BV([0,T],K)$ with distributional derivative u. Let $t \mapsto Y_t^v$ be the contraction propagator with initial time $s = 0$ associated with $\mathcal{A}_v(t) := e^{-v(t)B}Ae^{v(t)B}$. We define the *generalized propagator* associated with $A + u(t)B$ with initial time zero, to be $\Upsilon^{\mathfrak{d}v}_{t,0} = e^{v(t)B}Y_t^v$ for every t in $[0,T]$ and v in $BV([0,T],K)$ such that $v' = u$ in the distributional sense.

Proposition 13. *Let (A, B, K) satisfy Assumption 2 with A and B skew-adjoint. Let $v \in BV([0,T],K)$ continuous. Then, for every ψ_0 in $D(A)$, for every t in $[0,T]$,*

$$\Upsilon^{\mathfrak{d}v}_{t,0}\psi_0 = e^{v(t)B}\psi_0 + \int_{s=0}^{t} e^{(v(t)-v(s))B}A\Upsilon^{\mathfrak{d}v}_{s,0}\psi_0 \mathrm{d}s.$$

Proof. Since ψ_0 belongs to $D(A)$, Theorem 5 guarantees that $t \mapsto Y_t^v\psi_0$ is a strong solution of $y'(t) = e^{-v(t)B}Ae^{v(t)B}y(t)$. That is, for every t,

$$Y_t^v\psi_0 = \psi_0 + \int_{s=0}^{t} e^{-v(s)B}Ae^{v(s)B}Y_t^v\psi_0\mathrm{d}s,$$

hence, multiplying by $e^{v(t)B}$,

$$\Upsilon^{\mathfrak{d}v}_{t,0}\psi_0 = e^{v(t)B}\psi_0 + \int_{s=0}^{t} e^{(v(t)-v(s))B}Ae^{v(s)B}Y_t^v\psi_0\mathrm{d}s = e^{v(t)B}\psi_0 + \int_{s=0}^{t} e^{(v(t)-v(s))B}A\Upsilon^{\mathfrak{d}v}_{s,0}\psi_0\mathrm{d}s,$$

which concludes the proof. $\square$

Remark 14. Let $u \in \mathcal{R}([0,T])$ and define $v_0(t) = u((0,t])$ the associated right-continuous cumulative function and let $v \in BV([0,T], \mathbf{R})$ be such that $v' = u$. Then $v - v_0$ is in $BV([0,T], \mathbf{R})$ and it is almost everywhere 0 since it is supported on the, at most countable, set where v is not right-continuous.

The propagator Y_t^u does not depend on the choice of v being right-continuous or not at its discontinuities. Indeed the set of point of discontinuity is negligible and a Duhamel formula provides the equality of the propagators. On the other hand, the factor e^{vB} depends crucially on this choice. This explains the notation $\Upsilon^{\mathfrak{d}v}$ instead of Υ^u.

The reason for introducing the notion of generalized propagator is that imposing any extra requirement on the choice of v will lead to loose the compactness provided by Helly's Selection Theorem. This will, for instance, make the presentation of the principle exposed in Section 1.1.2 more complicate.

Notice that for any v_1 and v_2 in $BV([0,T], K)$ with the same distributional derivative one has

$$\Upsilon^{\mathfrak{d}v_1} = e^{(v_1 - v_2)B} \Upsilon^{\mathfrak{d}v_2}.$$

Lemma 14. *Let (A, B, K_1) satisfy Assumption 1 and (A, B, K_2) satisfy Assumption 2. Let $u : [0,T] \mapsto K_1$ be of bounded total variation and $U(t) := \int_0^t u(s)\mathrm{d}s \in K_2$ for all t in $[0,T]$.*

Let Υ_t^u be the propagator associated with $A + u(t)B$ with initial time $s = 0$. Let $t \mapsto Y_t^u$ be the contraction propagator associated with $\mathcal{A}(t) := e^{-U(t)B} A e^{U(t)B}$ with initial time $s = 0$.

Then

$$\Upsilon_t^u = e^{U(t)B} Y^{u(t)} \left(= \Upsilon_t^{\mathfrak{d}U(t)} \right),$$

for every $t \in [0,T]$.

Proof. Let $\psi_0 \in D(A)$ and define the continuous function $\Psi : t \mapsto e^{-\int_0^t u(s)\mathrm{d}sB} \Upsilon_t^u(\psi_0)$. By Theorem 5, $\Psi(t) \in D(A)$ is strongly right differentiable in t with right derivative

$$-u(t+0)B\Psi(t) + e^{-U(t)B}(A + u(t+0)B)\Upsilon_t^u(\psi_0) = e^{-U(t)B} A e^{U(t)B} \Psi(t).$$

By uniqueness, see Theorem 5, $\Psi(t) = Y_t^u \psi_0$ for every $t \in [0,T]$. □

Proposition 15. *Let $(A, B, [0, +\infty))$ satisfy Assumption 1 and (A, B, K) satisfy Assumption 2. Then for every $\psi_0 \in \mathcal{H}$ and $t \in [0,T]$ the map $\Upsilon_t(\psi_0) : u \mapsto \Upsilon_t^u(\psi_0) \in \mathcal{H}$ admits a unique continuous extension on $\{u \in \mathcal{R}([0,T]) \mid u([0,T]) \in K, u \text{positive}\}$ denoted by $\Upsilon_t(\psi_0)$ and satisfying*

$$\Upsilon_t^u(\psi_0) = e^{u((0,t])B} Y_t^u(\psi_0), \quad \textit{for every } t \in [0,T]. \tag{3.2}$$

Proof. For every $u \in \mathcal{R}([0,T])$ positive with $u([0,T]) \in K$ let $(u_n)_{n \in \mathbf{N}}$ be a sequence of (right-continuous) positive piecewise constant functions on $[0,T]$ such that $\int_0^T u_n \in K$ converging to u and which existence is given by Lemma 12.

From Remark 12 and Corollary 11, for every $\psi_0 \in \mathcal{H}$, $Y_t^{u_n}(\psi_0)$ tends to $Y_t^u(\psi_0)$ as n tends to ∞. We set $\Upsilon_t^u(\psi_0) = e^{u((0,t])B} Y_t^u(\psi_0)$. Then $\Upsilon_t^{u_n}(\psi_0)$ tends to $\Upsilon_t^u(\psi_0)$ as n tends to ∞. The uniqueness of the extension is guaranteed by Lemma 14. □

Remark 15. With respect to Definition 4, Proposition 15 fixes the choice of antiderivative of u to the right-continuous one, accordingly to the arbitrary choice for the notion of convergence for Radon measures presented in Appendix A. Different choices would have led to different choices for the antiderivative of u. Qualitatively speaking, any choice provides the same results in the sequel.

Remark 16. The definition of propagator associated with positive Radon measures given in (3.2) can be extended to signed Radon measures provided that $(A, B, \mathbf{R})$ satisfies Assumption 1. Notice, however, that if $(A, B, \mathbf{R})$ satisfies Assumption 1 then B is necessarily symmetric.

In the case in which B is not symmetric the definition of propagator can be extended to signed Radon measures provided that (A, B, K) satisfies Assumption 2. The uniqueness of the continuous extension can be obtained if $(A, B-c, [0,\infty))$ and $(A, -B-c', [0,\infty))$ satisfy Assumption 1. Indeed consider $u \in BV([0,T], \mathbf{R})$ and split u in the difference of positive part $u^+ := \max\{u, 0\}$ and negative part $u^- := \max\{-u, 0\}$. Then $A(t) = A + u^+(t)(B-c) + u^-(t)(-B-c')$ satisfies Assumption 4.

Proposition 16. *Let (A, B) satisfy Assumption 3 and $D(A) \subset D(B)$. Then for every ψ_0 in $D(A)$, for every $u \in L^1([0,T], \mathbf{R})$, the map $t \mapsto \Upsilon_t^u(\psi_0)$ satisfies*

$$\int_{[0,T]} \langle f'(t), \Upsilon_t^u(\psi_0)\rangle \mathrm{d}t = \int_{[0,T]} \langle f(t), A\Upsilon_t^u(\psi_0)\rangle \mathrm{d}t + \int_{[0,T]} \langle f(t), B\Upsilon_t^u(\psi_0)\rangle u(t)\mathrm{d}t\,, \tag{3.3}$$

for every $f \in C_0^1([0,T], \mathcal{H})$.

A mapping $t \mapsto \Upsilon_t^u(\psi_0)$ satisfying (3.3) is called *weak solution* of (2.1) with initial condition ψ_0.

Proof. For every $u \in L^1([0,T])$ let $(u_n)_{n\in\mathbf{N}}$ be a sequence of piecewise constant functions on $[0,T]$ that converges to u in $\mathcal{R}([0,T])$ (in the sense of Appendix A).

For every $f \in C_0^1([0,T], \mathcal{H})$,

$$-\int_{[0,T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0)\rangle \mathrm{d}t = \int_{[0,T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0)\rangle \mathrm{d}t + \int_{[0,T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0)\rangle u_n(t)\mathrm{d}t$$

since from Theorem 5, $Y_t^{u_n}(\psi_0) \in D(A)$ for any $t \in [0,T]$.

It is then sufficient to prove the following convergences

$$\lim_{n\to\infty} \int_{[0,T]} \langle f'(t), \Upsilon_t^{u_n}(\psi_0)\rangle \mathrm{d}t = \int_{[0,T]} \langle f'(t), \Upsilon_t^u(\psi_0)\rangle \mathrm{d}t, \tag{3.4}$$

$$\lim_{n\to\infty} \int_{[0,T]} \langle f(t), A\Upsilon_t^{u_n}(\psi_0)\rangle \mathrm{d}t = \int_{[0,T]} \langle f(t), A\Upsilon_t^u(\psi_0)\rangle \mathrm{d}t, \tag{3.5}$$

and

$$\lim_{n\to\infty} \int_{[0,T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0)\rangle u_n(t)\mathrm{d}t = \int_{[0,T]} \langle f(t), B\Upsilon_t^u(\psi_0)\rangle u(t)\mathrm{d}t. \tag{3.6}$$

Convergence (3.4) is a consequence of Lebesgue Dominated Convergence Theorem being the integrand uniformly bounded.

We can rewrite (3.5) as

$$\lim_{n\to\infty} \int_{[0,T]} \langle f(t), A\left(\Upsilon_t^{u_n}(\psi_0) - \Upsilon_t^u(\psi_0)\right)\rangle \mathrm{d}t = 0$$

Recall that the adjoint of A is also maximal dissipative as soon as A is maximal dissipative [TW09, Chapter 3.1]. We can then restrict to $f \in C_0^1([0,T], D(A^*))$ by replacing f with $\lambda(\lambda - A^*)^{-1}f$, where λ is a large positive since

$$t \in [0,T] \mapsto A\left(\Upsilon_t^{u_n}(\psi_0) - \Upsilon_t^u(\psi_0)\right) \in \mathcal{H},$$

is uniformly bounded, and conclude, as in (3.4), with Lebesgue Dominated Convergence Theorem.

The last convergence (3.6) reads

$$\begin{aligned}
&\int_{[0,T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0)\rangle u_n(t)\mathrm{d}t - \int_{[0,T]} \langle f(t), B\Upsilon_t^u(\psi_0)\rangle u(t)\mathrm{d}t \\
&= \int_{[0,T]} \langle f(t), B\Upsilon_t^{u_n}(\psi_0)\rangle (u_n - u)(t)\mathrm{d}t + \int_{[0,T]} \left(\langle f(t), B\Upsilon_t^{u_n}(\psi_0)\rangle - \langle f(t), B\Upsilon_t^u(\psi_0)\rangle\right) u(t)\mathrm{d}t. \qquad (3.7)
\end{aligned}$$

In order to prove the convergence for the second term of the right-hand side we have

$$\lim_{n\to\infty}\int_{[0,T]}\langle\left(B(1-A)^{-1}\right)^* f(t),(1-A)\left(\Upsilon_t^{u_n}(\psi_0)-\Upsilon_t^{u}(\psi_0)\right)\rangle\mathrm{d}t=0.$$

Indeed $B(1-A)^{-1}$ is bounded, so is its adjoint. The proof is then similar to (3.4) and (3.5).

Finally, from Theorem 5 and estimates (2.7), (2.8), and (2.9), there exists $C>0$ and $\omega>0$ depending on A and B only, such that

$$\begin{aligned}&\left|\int_{[0,T]}\langle f(t),B\Upsilon_t^{u_n}(\psi_0)\rangle(u-u_n)(t)\mathrm{d}t\right|\\&\le C\sup_{t\in[0,T]}\|f(t)\|\|B\|_A(1+CL_{[0,\|u\|_1]}(A,B)\|u\|_1)e^{2\omega\|u\|_1}\times\\&\times e^{(1+CL_{[0,\|u\|_1]}(A,B)\|u\|_1)e^{2\omega\|u\|_1}CL_{[0,\|u\|_1]}(A,B)\|u\|_1}\|\psi_0\|_{D(A)}\|u-u_n\|_1\\&\to_{n\to\infty}0\end{aligned}$$

since $f(0)=0$ and using Lemma 12 and $|u-u_n|=|u^+-u_n^+|+|u^--u_n^-|$ the sequence (u_n) converges to u in L^1-norm. □

Remark 17. An interesting question would be to understand the relation between the assumptions associated with the two constructions of propagators considered in this section. For example, on what extent does Assumption 3 ensure that $A+uB$ has a maximal dissipative closure for $u\in\mathbf{R}$?

This seems to be a hard question. However in the skew-adjoint case, the following considerations are in place. Let A and B be skew-adjoint with $D(A)\subset D(B)$. For any $\varphi_1\in\mathcal{H}$, any $\varphi_2\in D(A)$ the map

$$t\in K\mapsto\langle(1-\varepsilon A)^{-1}\varphi_1,e^{tB}Ae^{-tB}(1-\varepsilon A)^{-1}\varphi_2\rangle,$$

is Lipschitz, its distributional derivative is bounded uniformly in ε by the Banach-Steinhaus theorem. So that $[A,B]\in L(D(A)\cap D(B),(D(A)\cap D(B))^*)$ extends to an operator such that

$$[A,B]\in L(D(A),\mathcal{H}),$$

(with a slight abuse of notation we denote by $[A,B]$ the extension of $[A,B]$ to $L(D(A),\mathcal{H})$) and, similarly (using the same abuse of notation), for any $u\in\mathbf{R}$,

$$[A,A+uB]\in L(D(A),\mathcal{H}).$$

The Nelson commutator theorem, see [RS75, Section X.5], gives that $A+uB$ is essentially skew-adjoint for any $u\in\mathbf{R}$.

Remark 18. Considering Definition 2, $X(t,s)=e^{v(t)B}Y_{t,s}^u e^{-v(s)B}$ defines a propagator when v is continuous, that is when u has no atoms. Otherwise, we no longer require any continuity keeping in mind that v_0 the right-continuous cumulative function of u will lead to a right-continuous propagator which is compatible with the requirements on the initial conditions.

From Proposition 16, when v is absolutely continuous, $X(t,s)=e^{v(t)B}Y_{t,s}^u e^{-v(s)B}$ defines a weak solution of (2.1). The question whether or not it is possible to extend this proposition to Radon measures is then natural. If one considers, as in Section 5.2 below, $A=0$, B bounded, and $u=\delta_0$, then the solution of (2.1) is $1+H(t)B$, where H is a Heaviside function jumping at 0. Which is different from $e^{H(t)B}$ provided by our analysis.

Proposition 16 can be extended to measures with singular continuous part. Indeed any Radon measure is in the sequential closure of the set of absolute continuous measures for the convergence of sequences we consider (see Appendix A). Notice that in Lemma 12 the sequence is also narrow convergent. Since the propagators associated with absolute continuous and singular continuous measure are bounded continuous, the first term in (3.7) will tend to 0.

Nonexistence of bounded solution propagators for unbounded control potentials in the skew-adjoint case. Let us consider the possible extensions of Proposition 16 to the case of a pair of skew-adjoint operators (A, B). We exhibit here an example of system (2.1) with a Radon measure control for which it is not possible to construct a strong solution by applying a bounded propagator to the initial condition (even if this is in the domain of the generator).

Let $\psi_0 \in D(A)$ and $\psi_1 \in D(B)$ with $B\psi_1 \in D(A)$ then for any solution of (2.1) for $u = \delta_{T/2}$ with initial condition ψ_0 at $t = 0$ the jump at $T/2$ is exactly $B\psi(T/2)$ (after integration of (2.1) around $T/2$). So, setting $\psi(T/2) = \psi_1$, we have

$$\Upsilon_t^u(\psi_0) = \begin{cases} e^{tA}\psi_0 & \text{for } t \in [0, \frac{T}{2}), \\ \psi_1 & \text{for } t = \frac{T}{2}, \\ e^{tA}\psi_0 + e^{(t-\frac{T}{2})A}B\psi_1 & \text{for } t \in (\frac{T}{2}, T]. \end{cases}$$

The determination of ψ_1 leads to a uniqueness issue and a modelling interpretation. A way to overcome this issue is to impose a continuity at $t = T/2$. Note that the left-continuity leads to

$$\psi_1 = e^{\frac{T}{2}A}\psi_0 + B\psi_1.$$

So if 1 is in the spectrum of B then for some ψ_0 this is not solvable. Note that with $u = \alpha\delta_{T/2}$ the problem is the same for every α in the spectrum of B. This excludes the possibility to construct a left-continuous propagator.

A natural requirement seems to be the right-continuity and thus $\psi_1 = e^{\frac{T}{2}A}\psi_0$. Then, when B is unbounded, the issue is to extend continuously the propagator from $D(A)$ to $\mathcal{H}$ as for some $\psi_0 \in \mathcal{H}$, one may have that $e^{\frac{T}{2}A}\psi_0 \notin D(B)$.

By convexity, a linear combination of left and right continuity will lead to the same kind of contradictions.

In conclusion, when B is unbounded one cannot expect to construct bounded solutions with Radon controls in the skew-adjoint case. In Section 5, we prove that when B is bounded, there exists a strongly regulated propagator defining a weak solution of (2.1). This propagator is not necessarily a contraction.

Similar questions arise for ODEs. We refer for instance to [PD82].

Nonetheless, with absolutely continuous Radon measures, we have built proper propagators and the extension to Radon measures presented in this work has consequences in the analysis of the attainable sets as presented in the sequel.

3.3 The attainable set

The key result in the proof of Theorem 1 is given by the following proposition.

Proposition 17. *Let $T > 0$. Let $\psi_0 \in \mathcal{H}$. Let (A, B) satisfy Assumption 3. Then, for every $L > 0$, the set*

$$\{\Upsilon_t^u(\psi_0) \,:\, u \in \mathcal{R}([0, T]), |u|((0, T]) \leq L, t \in [0, T]\},$$

is relatively compact in $\mathcal{H}$.

Proof. Consider a sequence $(u_n)_{n\in\mathbf{N}} \subset \mathcal{R}([0, T])$ such that $|u_n|((0, T]) \leq L$ for every n. By Helly's Selection Theorem the sequence $v_n : t \mapsto u_n((0, t])$, has a subsequence pointwise converging to some $v \in BV([0, T])$, $\|v\| \leq L$. We relabel this convergent subsequence by $(v_n)_{n\in\mathbf{N}}$.

From (2.9) we have that $\mathcal{A}_n(t) = e^{-u_n((0,t])B}Ae^{u_n((0,t])B}$ is uniformly bounded in $BV([0, T], L(\mathcal{D}, \mathcal{H}))$. By Corollary 11, $t \mapsto Y_t^{v_n}(\psi_0)$ converges uniformly on $[0, T]$ to $t \mapsto Y_t^v(\psi_0)$ as $n \to \infty$. For any sequence $(t_n)_n$, $(v_n((0, t_n]))_n$ is a bounded sequence. In particular, it has a strongly convergent subsequence and so is $(e^{v_n((0,t_n])B})_n$. ∎

Remark 19. Note that the set $\{\Upsilon^u(\psi_0) \,:\, u \in L^1([0,T],\mathbf{R}), \|u\|_{L^1} \leq L\}$ is relatively compact in $L^\infty([0,T],\mathcal{H})$. However, despite the compactness of $[0,T]$, the set $\{\Upsilon^u(\psi_0) \,:\, u \in \mathcal{R}([0,T]), |u|((0,T]) \leq L\}$ may be not relatively compact in $L^\infty([0,T],\mathcal{H})$. Indeed, if this set were relatively compact, then the *generalized propagator* associated with $A+u(t)B$ would be strongly continuous, due to the point-wise density of solutions of (2.1) which are continuous. This is not the case in general due to the factor $e^{u((0,t])B}$ in (3.2).

From the above result the attainable set is a countable union of totally bounded sets.

Corollary 18. *Let $\psi_0 \in \mathcal{H}$. If (A,B) satisfies Assumption 3 then*

$$\mathcal{A}tt_{\mathcal{R}}(\psi_0) := \{\Upsilon^u_t(\psi_0), u \in \mathcal{R}([0,+\infty)), t \geq 0\}$$

is contained in a countable union of compact subsets of $\mathcal{H}$.

Proof. The attainable set can be rewritten as

$$\bigcup_{L,T>0} \{\Upsilon^u_t(\psi_0), u \in \mathcal{R}([0,T]), |u|((0,T]) \leq L, t \in [0,T]\}$$

and this union can be, in fact, restricted to L,T in a countable set, for instance $\mathbf{N}^2$. Then Proposition 17 tells that each set of the union is relatively compact in $\mathcal{H}$ and thus with empty interior. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. The well-posedness result for L^1 controls is a consequence of Proposition 16 proved for Radon controls. The conclusion on the attainable set for L^1 controls is a consequence of Corollary 18 proved for Radon controls. $\square$

4 Higher order norm estimates for mildly coupled systems

In the following we will restrict our analysis to the skew-adjoint case. The motivation for this assumption is twofold. On the one hand this is the case for most of the mathematical objects appearing in quantum mechanics and, on the other hand, the restriction to skew-adjoint operators makes the analysis simpler.

The aim of this section is to analyze under which conditions the solution built in the previous sections are smooth in the scale of A. This is indeed the rationale for stating assumptions 1, 2, or 3 in $D(|A|^{k/2})$ instead of $\mathcal{H}$. Our aim is to provide a somewhat simpler criteria showing that the extension of assumptions on B will be sufficient. To this aim, the A-boundedness of B as an operator acting on $D(|A|^{k/2})$ is crucial and it is stated in Lemma 23 below which is the cornerstone of the analysis of this section. This is especially important if we want to obtain the regularity of propagators in the scale of A up to the order $k/2$. For lower orders, a simple interpolation argument provides the desired results. The criteria will be used in a perturbative framework (Kato-Rellich type argument) and we will not consider the entire set K for the values of u, unless we assume that the domain of powers of $A+uB$ are the same for any $u \in K$. We recall that in the dissipative framework in order to use Kato–Rellich criterion u has to be nonnegative when B is dissipative, below we assume that both B and $-B$ have dissipativity properties (up to a shift by a constant as in Assumption (A2.1) or (A3.2)) so that the sign of u does not play any role.

This shows that for time reversible systems, the input-output mapping does not change the regularity with respect to A in the spirit of Section 1.1.3. Since eigenvectors belong to any $D(|A|^k)$ this shows that exact controllability clearly relies on the regularity of B in the scale of A.

4.1 The mild coupling

Given a skew-adjoint operator A and $k \in \mathbf{R}$, $k \geq 0$, we define

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k \psi, \psi \rangle}.$$

Definition 5 (Mild coupling)**.** Let k be a nonnegative real. A pair of *skew-adjoint* operators (A, B) is *k-mildly coupled* if

(i) A is invertible with bounded inverse from $D(A)$ to $\mathcal{H}$,

(ii) for any real t, $e^{tB} D(|A|^{k/2}) \subset D(|A|^{k/2})$,

(iii) there exists $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\| \cdot \|_{k/2}$.

The *optimal exponential growth*, the growth bound of the semigroups generated by $\pm B$, is defined by

$$c_k(A, B) := \sup_{t \in \mathbf{R}} \frac{\log \|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2})}}{|t|}. \tag{4.1}$$

Remark 20. As in Section 5 below, if

$$t \mapsto e^{tB}$$

is a strongly continuous semigroup on $D(|A|^{k/2})$ then there exists $\omega > 0$ and $C > 0$ such that

$$\|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))} \leq C e^{\omega t}, \quad \text{for all } t > 0.$$

The invertibility of A is needed to ensure that $\| \cdot \|_{k/2}$ is a norm equivalent to the graph norm of $D(|A|^{k/2})$. The use of the associated norm is due to the interpolation criterion used in Lemma 22 below.

Remark 21. The quantity $c_k(A, B)$ is related to the growth abscissa of B in $D(|A|^{k/2})$. The link between the growth abscissa and the spectral radius of a semigroup on a Hilbert space is considered in [Prü84, Section 3].

Remark 22. For many systems encountered in the physics literature, the operator A is skew-adjoint with a spectral gap. Hence the invertibility of A can be obtained by replacing A by $A - \lambda \mathrm{i}$ for a suitable λ in $\mathbf{R}$. Notice that this translation on A only induces a global phase shift on the propagator that is physically irrelevant (i.e., undetectable by observations).

The following proposition gives another characterization of mild coupling using Hille–Yosida Theorem.

Proposition 19. *Let k be a nonnegative real. A pair of skew-adjoint operators (A, B) with A invertible is k-*mildly coupled* if and only if B is closed in $D(|A|^{k/2})$, and there exists ω such that*

$$\|(\lambda I - B)^{-1}\|_{L(D(|A|^{k/2}), D(|A|^{k/2})} \leq \frac{1}{|\lambda| - \omega}, \tag{4.2}$$

for every real λ, $|\lambda| > \omega$ in the resolvent set of B.

Moreover the smallest ω satisfying (4.2) *is $c_k(A, B)$ given by* (4.1).

Proof. If (A, B) be *k-mildly coupled* then $B - c_k(A, B)$ is the generator of a contraction semigroup in $D(|A|^{k/2})$. From Hille–Yosida Theorem, we deduce the equivalence with Definition 5. □

The following proposition gives an equivalent definition which may be easier to check in practice.

Proposition 20. *Let k be a nonnegative real. A pair of skew-adjoint operators (A,B) with A invertible is k-mildly coupled, if and only if for some $\omega > 0$,*

$$(\omega \pm B)^{-1} D(|A|^{k/2}) \subset D(|A|^{k/2})$$

and for any $\psi \in (\omega - B)^{-1} D(|A|^{k/2}) = (\omega + B)^{-1} D(|A|^{k/2})$, one has

$$|\Re\langle |A|^k \psi, B\psi\rangle| \leq \omega \|\psi\|^2_{D(|A|^{k/2})}. \tag{4.3}$$

Moreover the smallest ω satisfying (4.3) is $c_k(A,B)$ given by (4.1).

Proof. We first notice that, for any ω in the resolvent sets of B and $-B$,

$$(\omega \pm B)^{-1} D(|A|^{k/2}) \subset D(|A|^{k/2})$$

implies $(\omega - B)^{-1} D(|A|^{k/2}) = (\omega + B)^{-1} D(|A|^{k/2})$. Indeed, from the resolvent identity, we deduce

$$(\omega - B)^{-1} D(|A|^{k/2}) \subset (\omega + B)^{-1} D(|A|^{k/2}).$$

Assume that (A,B) is k-mildly coupled, then since $B - c_k(A,B)$ and $-B - c_k(A,B)$ are generator of contraction semigroups on $D(|A|^{k/2})$, they are closed and maximal dissipative on $D(|A|^{k/2})$, their respective resolvent sets contains positive half lines (by means of Hille–Yosida theorem) and their domain, by definition of resolvent, is $(\omega \pm B)^{-1} D(|A|^{k/2})$ for any $\omega > c_k(A,B)$. Since they are maximal dissipative, we have that

$$|\Re\langle |A|^k \psi, B\psi\rangle| \leq c_k(A,B) \|\psi\|^2_{D(|A|^{k/2})}.$$

for any $\psi \in (\omega - B)^{-1} D(|A|^{k/2}) = (\omega + B)^{-1} D(|A|^{k/2})$.

Reciprocally, $B + \omega$ and $-B + \omega$ are closed as operators on $\mathcal{H}$ and so they are closed on $D(|A|^{k/2})$. Since $B + \omega$ and $-B + \omega$ are dissipative on $D(|A|^{k/2})$, they are generators of a contractions semigroups if they are surjective. So they are since $(\pm B + \omega)^{-1} f \in D(|A|^{k/2})$ for any $f \in D(|A|^{k/2})$. ∎

The notion of mild coupling is related to the notion of "weak coupling" introduced in [BCC13]. The relation between these two definitions is given by the following lemma.

Lemma 21. *Let (A,B) be a pair of linear operators such that A is invertible and skew-adjoint with domain $D(A)$, B is skew-symmetric with $D(A) \subset D(B)$, $A + uB$ (seen as an operator acting on $\mathcal{H}$) is essentially skew-adjoint on $D(A)$ for every u in $\mathbf{R}$, $D(|A + uB|^{k/2}) = D(|A|^{k/2})$ for some $k \geq 1$ and for any real u, and there exists a constant C such that for every ψ in $D(|A|^k)$,*

$$|\Re\langle |A|^k \psi, B\psi\rangle| \leq C |\langle |A|^k \psi, \psi\rangle|.$$

Then (A,B) is k-mildly coupled and $c_k(A,B)$ is the best possible constant C in the above inequality.

Proof. The assumption that there exists $k \geq 1$ and a constant C such that for every ψ in $D(|A|^k)$,

$$|\Re\langle |A|^k \psi, B\psi\rangle| \leq C |\langle |A|^k \psi, \psi\rangle|$$

and the Nelson Commutator Theorem, see [RS75, Section X.5], imply that B is essentially skew-adjoint on the domain $D(|A|^{k/2})$. Therefore B is essentially skew-adjoint on $D(A)$. Then Trotter Product Formula, see [RS72, Theorem VIII.31], implies that

$$\left(e^{\frac{t}{n}(A+uB)} e^{-\frac{t}{n}A}\right)^n \to e^{tuB}$$

in the strong sense as n goes to infinity. Since each of the term of the above sequence is bounded on $D(|A|^{k/2})$ with a bound $e^{C|t||u|}$, see [BCC13, Proposition 2], we conclude that e^{tB} is bounded on $D(|A|^{k/2})$ with the same bound $e^{C|t||u|}$. Then (A,B) is k-mildly coupled. ∎

Remark 23. In general, (A, B) can be k-mildly coupled without being weakly coupled (in the sense of [BCC13, Definition 1]) or without satisfying the assumption of Lemma 21. Indeed for any invertible skew-adjoint unbounded operator $(A, \mathrm{i}A^2)$ is 2-mildly coupled and $D(A) \not\subset D(\mathrm{i}A^2)$ or $D(A + \mathrm{i}A^2) = D(A^2) \neq D(A)$.

Let us state state an interpolation result.

Lemma 22. *Let k be a positive real. If (A, B) is k-mildly coupled then (A, B) is s-mildly coupled for any $s \in [0, k]$ and*

$$c_s(A, B) \leq \frac{s}{k} c_k(A, B).$$

Proof. We will consider $s \in (0, k)$. Indeed, for $s = k$ this is obvious and $s = 0$ there is nothing to prove since B is skew-adjoint by assumption.

Moreover since B is skew-adjoint, for every ψ in $D(|A|^{\frac{k}{2}})$,

$$\|e^{tB}\psi\|_{D(|A|^{k/2})} = \||A|^{k/2} e^{tB}\psi\| = \||\mathcal{A}(t)|^{k/2}\psi\|.$$

where $\mathcal{A}(t) = e^{-tB} A e^{tB}$ (which is skew-adjoint with domain $D(A)$).

Since (A, B) is k-mildly coupled we deduce

$$\frac{1}{\|A^{-1}\|^k} \leq |\mathcal{A}(t)|^k \leq e^{2c|t|} |A|^k.$$

which from Proposition 39 in Appendix B yields

$$|\mathcal{A}(t)|^s \leq e^{2cs|t|/k} |A|^s.$$

This concludes the proof. $\square$

A corollary of this interpolation result is the following result which is crucial in our analysis. It shows that if (A, B) is k-mildly coupled the A-boundedness of B extends naturally to $D(|A|^{k/2})$. Hence, from now on, we will work in $D(|A|^{k/2})$, that is we consider $\mathcal{H} = D(|A|^{k/2})$.

Lemma 23. *Let k be a nonnegative real. Let (A, B) be k-mildly coupled and such that B is A-bounded. Then*

$$\inf_{\lambda > 0} \|B(A - \lambda)^{-1}\|_{L(D(|A|^{\frac{k}{2}}), D(|A|^{\frac{k}{2}}))} \leq \|B\|_A$$

Proof. Denote by $\mathcal{H}_s = D(|A|^s)$, endowed with the graph norm, and by $\mathcal{H}_{-s} = D(|A|^s)^*$, for $s \in [0, +\infty]$. Note that, if $s' \leq s$, $\mathcal{H}_s \subset \mathcal{H}_{s'}$ with continuous embeddings. The domain of $|A|^\sigma$, $\sigma \geq 0$, in $\mathcal{H}_s$ is $\mathcal{H}_{s+\sigma}$ and $|A|^\sigma \mathcal{H}_s = \mathcal{H}_{s-\sigma}$ as A is invertible. The real interpolation $(\mathcal{H}_{s_0}, \mathcal{H}_{s_1})_{\theta,2}$ is $\mathcal{H}_{s_\theta}$ with $s_\theta = \theta s_1 + (1 - \theta) s_0$. See, for instance, [ABG96, Section 2.8].

The proof follows [RS75, Section X.5]. The commutator $[|A|^k, B] = |A|^k B - B|A|^k$ is defined from $\mathcal{H}_{k+1}$ to $\mathcal{H}_{-k}$ and since B is k-mildly coupled, for $\psi \in \mathcal{H}_{k+1}$, we have

$$\left| \left\langle \psi, (|A|^k B - B|A|^k)\psi \right\rangle \right| = 2 \left| \Re \left\langle B\psi, |A|^k \psi \right\rangle \right| \leq 2c_k(A, B) \||A|^{k/2}\psi\|^2.$$

This provides, after polarization, the boundedness of $[|A|^k, B]$ from $\mathcal{H}_{k/2}$ to $\mathcal{H}_{-k/2}$. For any s in $[0, k]$ due to Lemma 22, B is s-mildly coupled. Hence the commutator $[|A|^s, B] = |A|^s B - B|A|^s$ extends as a bounded operator from $\mathcal{H}_{s/2}$ to $\mathcal{H}_{-s/2}$ for any s in $[0, k]$.

Recall B is A-bounded, that is B bounded from $\mathcal{H}_1$ to $\mathcal{H}$. As $-B^*$ is an extension of B and is bounded from $\mathcal{H}$ to $\mathcal{H}_{-1}$, with same norm, by interpolation, B is bounded from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$ for any

$s \in [-1, 0]$. The bound on the norm of B as an operator from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$, $s \in [-1, 0]$ is thus smaller than the norm of B as an operator from $\mathcal{H}_1$ to $\mathcal{H}$. Hence, we have

$$\inf_{\lambda>0} \|B(A-\lambda)^{-1}\|_{L(\mathcal{H}_s,\mathcal{H}_s)} \leq \|B\|_A.$$

Therefore, for $k \in [0, 2]$, $B|A|^k$ is bounded from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{-k/2}$. As $|A|^k B = B|A|^k + [|A|^k, B]$, $|A|^k B$ extends as a bounded operator from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{-k/2}$ and B is bounded from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{k/2}$. Hence B is bounded from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$ for any $s \in [-1, k/2]$. As for the norm, for any ϵ positive, there exists Λ_ϵ such that, for any $\psi \in \mathcal{H}_{k/2+1}$,

$$\begin{aligned}
\|B\psi\|_{k/2} &= \||A|^{-k/2}|A|^k B\psi\| \\
&\leq \||A|^{-k/2}B|A|^k\psi\| + \||A|^{-k/2}[|A|^k, B]\psi\| \\
&\leq (\|B\|_A + \epsilon)\||A|^{k/2}|A + \Lambda_\epsilon|\psi\| + 2c_k(A,B)\||A|^{k/2}\psi\| \\
&\leq (\|B\|_A + \epsilon)\|A|^{k/2}|A + L_\epsilon|\psi\|
\end{aligned}$$

for $L_\epsilon \geq \Lambda_\epsilon$ large enough. Then, we deduce

$$\inf_{\lambda>0} \|B(A-\lambda)^{-1}\|_{L(\mathcal{H}_{k/2},\mathcal{H}_{k/2})} \leq \|B\|_A.$$

Hence the Lemma is proved for k in $[0, 2]$. Assume that $k \geq 2$, we now extend the lemma to k by an induction. Assume that B is bounded from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$ for any $s \in [-1, (k-2)/2]$. So B is bounded from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$ for any $s \in [-k/2, 0]$ by duality.

We have $B|A|^k$ is bounded from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{-k/2}$. As $|A|^k B = B|A|^k + [|A|^k, B]$, $|A|^k B$ extends as a bounded operator from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{-k/2}$, and B is bounded from $\mathcal{H}_{k/2+1}$ to $\mathcal{H}_{k/2}$. Hence B is bounded from $\mathcal{H}_{1+s}$ to $\mathcal{H}_s$ for any $s \in [-1, k/2]$. This concludes the induction since the norm estimate is obtained as in the initialisation. □

The following result shows sufficient conditions to have $D(|A + uB|^{k/2}) = D(|A|^{k/2})$ as required in Lemma 21. Checking this property in practice may be a hard task in general. Recall that as $D(A) \subset D(B)$, $A + uB$ is self-adjoint with $D(A + uB) = D(A)$ for sufficiently small u by Kato–Rellich theorem.

Lemma 24. *Let k be a positive real, (A, B) be k-mildly coupled, and $u \in \mathbf{R}$ such that $|u| < 1/\|B\|_A$. Then $D(|A|^s) = D(|A + uB|^s)$ for every $s \in [0, k/2 + 1]$.*

Proof. We proceed by induction on j to prove $D(|A|^{k/2-\lfloor k/2\rfloor+j}) = D(|A + uB|^{k/2-\lfloor k/2\rfloor+j})$ for $j \leq \lfloor k/2\rfloor + 1$. By Kato–Rellich theorem, $D(A) = D(A + uB)$ for every u in $(-1/\|B\|_A, 1/\|B\|_A)$. By interpolation, see Corollary 43 in Appendix B, $D(|A|^s) = D(|A+uB|^s)$ for $0 \leq s \leq 1$ and in particular for $s = \frac{k}{2} - \lfloor \frac{k}{2} \rfloor$. This initializes the induction for $j = 0$.

Let us assume that $D(A^\ell) = D((A + uB)^\ell)$ for some $\ell \leq \lfloor k/2\rfloor$. By definition,

$$D(A^{\ell+1}) = \{f \in D(A^\ell) | Af \in D(A^\ell)\},$$

and, using the inductive hypothesis,

$$D(|A+uB|^{\ell+1}) = \{f \in D(|A+uB|^\ell) | |A+uB|f \in D(|A+uB|^\ell)\} = \{f \in D(|A|^\ell) | (A+uB)f \in D(|A|^\ell)\}.$$

So that $D(|A + uB|^{\ell+1})$ is the domain of $A + uB$ as an operator acting on $D(|A|^\ell)$. The domain of A as an operator acting on $D(|A|^\ell)$ is $D(|A|^{\ell+1})$. Since A is skew adjoint on $D(|A|^\ell)$ and $B - c$ (or $-B - c'$) is dissipative, since $\ell \leq k/2$, in $D(|A|^\ell)$ due to Proposition 20, using Lemma 23 and Kato-Rellich theorem we conclude that $A + uB - c''$ with domain $D(|A|^{\ell+1})$ is maximal dissipative in $D(|A|^\ell)$ for some constant c'' sufficiently large. This implies $D(|A + uB|^{\ell+1}) = D(|A|^{\ell+1})$.

This completes the iteration and provides the conclusion. □

4.2 Higher regularity

From Lemma 6 and Proposition 7, we deduce the following statement.

Proposition 25. *Let k be a nonnegative real, (A,B) be k-mildly coupled, B be A-bounded, and $K = [-1/(2\|B\|_A), 1/(2\|B\|_A)]$. For any $u \in BV([0,T],K)$ consider the family of contraction propagators Υ^u associated with $A+u(t)B$. Then $\Upsilon^u_{t,s}(D(|A|^{k/2})) \subset D(|A|^{k/2})$, for any $(s,t) \in \Delta_{[0,T]}$, and:*

(i) for any $t \in [0,T]$ and for any $\psi_0 \in D(|A|^{k/2})$

$$\|\Upsilon^u_t(\psi_0)\|_{k/2} \leq e^{c_k(A,B)\int_0^t |u|}\|\psi_0\|_{k/2}.$$

(ii) for any $t \in [0,T]$ and for any $\psi_0 \in D(|A|^{1+k/2})$ there exists M (depending only on A, B, and $\|u\|_{L^\infty([0,T])}$) such that

$$\|\Upsilon^u_t(\psi_0)\|_{1+k/2} \leq M e^{M\mathrm{TV}(u,([0,t],K))} e^{c_k(A,B)\int_0^t |u|}\|\psi_0\|_{1+k/2}.$$

Moreover, for every ε in $(0, 1+k/2)$, for every ψ_0 in $D(|A|^{k/2+1-\varepsilon})$, the end-point mapping

$$\begin{aligned} \Upsilon_T(\psi_0) : BV([0,T],K) &\to D(|A|^{k/2+1-\varepsilon}) \\ u &\mapsto \Upsilon^u_T(\psi_0) \end{aligned}$$

is continuous.

Proof. Let us begin with the case $k=0$. By hypothesis, (A,B,K) satisfies Assumption 1 and, by Lemma 6, $t \mapsto A+u(t)B$ satisfies Assumption 4 for every u in $BV(I,K)$. The statements (i) and (ii) for $k=0$ follow from Theorem 5. The continuity of the end-point mapping with value in $\mathcal{H}$ follows from Corollary 9 and item (ii).

The idea to deal with the case $k>0$ is then to prove the existence of propagator in $D(|A|^{k/2})$. By Lemma 22, this implies existence of a propagator in $D(|A|^{s/2})$ for any $s \in [0,k]$. Since $D(|A|^{k/2}) \subset \mathcal{H}$, by uniqueness, Theorem 5, each propagator is the restriction of the one defined for $s=0$.

Consider $k>0$. If $c_k(A,B)=0$, Lemma 23 ensures that the triple (A,B,K) satisfies Assumption 1 in $D(|A|^{\frac{k}{2}})$ and, for any u in $BV(I,K)$, the mapping $t \mapsto A+u(t)B$ satisfies Assumption 4 in $D(|A|^{\frac{k}{2}})$. Statements (i) and (ii) follows from Theorem 5. In the case where $c_k(A,B)>0$, in order to obtain contraction semigroups, we consider $A(t) = A+u(t)B - c_k(A,B)|u(t)|$. This induces minor technical variations in the proof to check that $t \mapsto A(t)$ satisfies Assumption 4. For the reader's sake, we detail them below.

Using (A,B) to be k-mildly coupled, in Lemma 23 and Kato–Rellich theorem for dissipative operators (see [RS78, Corollary of Theorem X.50]) provides that $A(t)$ satisfies Assumption (A4.1) with $I=[0,T]$ and $D(|A|^{\frac{k}{2}})$ instead of $\mathcal{H}$. Notice that indeed the domain of A as an operator acting on $D(|A|^{\frac{k}{2}})$ is $D(|A|^{1+\frac{k}{2}})$. In the following, we check Assumptions (A4.2) and (A4.3).

From Lemma 23, if $a \in (\|B\|_A, 2\|B\|_A)$ we deduce that there exists b_a such that for any $\psi \in D(|A|^{1+k/2})$

$$\begin{aligned} \|(1-A-uB+c_k(A,B)|u|)\psi\|_{k/2} \geq & \quad \|(1-A)\psi\|_{k/2} - |u|\|B\psi\|_{k/2} - c_k(A,B)|u|\|\psi\|_{k/2} \\ \geq & \quad (1-a|u|)\|(1-A)\psi\|_{k/2} - |u|(b_a + c_k(A,B))\|\psi\|_{k/2} \end{aligned}$$

or

$$\|(1-A-uB+c_k(A,B)|u|)\psi\|_{k/2} + |u|(b_a+c_k(A,B))\|\psi\|_{k/2} \geq (1-a|u|)\|(1-A)\psi\|_{k/2}.$$

Note that for the choice of K and a we have that $a|u| < 1$. Therefore

$$\|B\psi\|_{k/2} \leq a\|(1-A)\psi\|_{k/2} \leq \frac{a}{1-a|u|}\|(1-A-uB+c_k(A,B)|u|)\psi\|_{k/2} + \frac{b_a+c_k(A,B)}{1-a|u|}\|\psi\|_{k/2}.$$

Hence,

$$\begin{aligned}\|(1-A(t))^{-1}\|_{L(D(|A|^{k/2}),D(|A|^{1+k/2}))} &= \|A(1-A-u(t)B+c_k(A,B)|u(t)|)^{-1}\|_{k/2} \\ &\leq \|(A+u(t)B-c_k(A,B)|u(t)|)(1-A-u(t)B+c_k(A,B)|u(t)|)^{-1}\|_{k/2} \\ &\quad + |u(t)|\|B(1-A-u(t)B+c_k(A,B)|u(t)|)^{-1}\|_{k/2} \\ &\leq 2 + \frac{|u(t)|}{1-a|u(t)|}\left(a+b_a+c_k(A,B)\right).\end{aligned}$$

Recall that, by assumption, $\sup_{t\in[0,T]} a|u(t)| \leq \frac{a}{2\|B\|_A} < 1$. Taking the supremum on $t \in [0,T]$ leads to

$$\sup_{t\in[0,T]} \|(1-A(t))^{-1}\|_{L(D(|A|^{k/2}),D(|A|^{1+k/2}))} \leq 2 + \frac{a}{2\|B\|_A - a}\left(a+b_a+c_k(A,B)\right). \tag{4.4}$$

As moreover for $A_n(t) = A + u_n(t)B - |u_n(t)|c_k(A,B)$ and $A(t) = A + u(t)B - |u(t)|c_k(A,B)$ and λ sufficiently large such that $\mathrm{TV}(A_n, ([0,T], L(D(A),\mathcal{H}))) \leq \mathrm{TV}(u_n, ([0,T],K))(\|B\|_{L(D(A),\mathcal{H})} + c_k(A,B))$, $\|A_n(0)\|_{L(D(A),\mathcal{H}))} \leq 1 + |u_n(0)|(\|B\|_{L(D(A),\mathcal{H})} + c_k(A,B))$, and

$$\begin{aligned}(A_n(t)-\lambda)^{-1} - (A(t)-\lambda)^{-1} =& (u_n(t)-u(t))(A_n(t)-\lambda)^{-1}B(A(t)-\lambda)^{-1} \\ &+ (|u_n(t)|-|u(t)|)(A_n(t)-\lambda)^{-1}c_k(A,B)(A(t)-\lambda)^{-1}\end{aligned}$$

so that the strong resolvent convergence of A_n to A turns to be a consequence of the convergence of u_n to u in $BV([0,T],K)$. □

Remark 24. The bound on the control $|u| \leq 1/(2\|B\|_A)$ in Proposition 25 is technical. We could enlarge the set of admissible control and consider $K = [-1/\|B\|_A + \varepsilon, 1/\|B\|_A - \varepsilon]$ for some $\varepsilon > 0$. In this case the constant a in the proof would be in the open interval $(\|B\|_A, \|B\|_A/(1-\varepsilon\|B\|_A))$, the bound (4.4) would depend on ε, and would tend to infinity as ε goes to 0.

We now state another version of Corollary 18.

Corollary 26. *Let k be a nonnegative real. Let (A,B) be k-mildly coupled, B be A-bounded, and $K = (-1/(2\|B\|_A), 1/(2\|B\|_A))$. Then, for every ε in $(0, 1+k/2)$ and every ψ_0 in $D(|A|^{1+k/2-\varepsilon})$ $\{\ \Upsilon^u_t(\psi_0), u \in BV([0,+\infty),K), t \geq 0\}$ is a countable union of relatively compact subsets in $D(|A|^{\frac{k}{2}+1-\varepsilon})$.*

Proof. The proof follows step-by-step the principle exposed in Section 1.1.2 and the proof of Corollary 10. □

Proof of Theorem 3. Theorem 3 is consequence of Corollary 26 when $\|B\|_A$ vanishes. □

Remarks on the exact controllability associated with the time reversibility. Let (A,B,K) satisfies Assumption 1 (or Assumptions 2) with A skew-adjoint and B skew-symmetric then $(-A,-B,K)$ satisfies Assumption 1 (or Assumptions 2). If (A,B) is k-mildly coupled then $(-A,-B)$ is k-mildly coupled.

For u, a bounded variation function (or a Radon measure, see Section 4.3 below) on $(0,T]$ with value in K and Υ^u the associated contraction propagator. For any $(t,s) \in \Delta_{[0,T]}$, $\Upsilon^u_{t,s}$ is unitary and its inverse coincides with $\Upsilon^{u(T-\cdot)}_{T-s,T-t}$ where $u(T-\cdot)$ denotes $t \in [0,T] \mapsto u(T-t)$ in the framework of Assumption 1 (or $t \in [0,T] \mapsto u((0,T]) - u((0,t]) = u([t,T))$ in the framework of Assumption 2).

4.3 Extension to Radon measures

The conclusion of Proposition 17 can be extended to $D(|A|^{k/2})$ if Assumption (A3.3) holds true in $D(|A|^{\frac{k}{2}})$ instead of $\mathcal{H}$. This is indeed the only missing assumption needed in order to apply Corollary 11 with $D(|A|^{\frac{k}{2}})$ instead of $\mathcal{H}$. Without this assumption the following result together with the interpolation result of Lemma 38 gives an interesting extension.

Proposition 27. *Let k be a positive real. Let (A, B) satisfy Assumption 3 and be k-mildly coupled. Then, for every $s \in [0, k]$, $\psi_0 \in D(|A|^{s/2})$, for every $T \geq 0$, one has $\Upsilon_T^{\mathfrak{d}v}(\psi_0) \in D(|A|^{s/2})$ and*

$$\|\Upsilon_T^{\mathfrak{d}v}(\psi_0)\|_{s/2} \leq e^{\frac{s}{k}c_k(A,B)|u|([0,T])}\|\psi_0\|_{s/2}$$

for every v in $BV([0, T], K)$ with derivative $v' = u \in \mathcal{R}([0, T])$.

Proof. We give the proof for $s = k$, then by Lemma 22 the proof applies to the case $s < k$.

Consider a sequence v_n of piecewise constant functions converging to v pointwise with $\|v_n\|_{BV([0,T])} \leq K$. Then v_n is the cumulative function of v'_n, a discrete sum of Dirac delta functions and, from (3.2), $\Upsilon_t^{\mathfrak{d}v_n}$ is a product of unitary operators of the form

$$e^{vB}e^{-vB}e^{tA}e^{vB} = e^{tA}e^{vB}.$$

So that, for every ψ in $D(|A|^{k/2})$,

$$\|e^{vB}e^{-vB}e^{tA}e^{vB}\psi\|_{k/2} = \|e^{vB}\psi\|_{k/2} \leq M(v)\|\psi\|_{k/2}$$

where $M(v) := \|e^{vB}\|_{L(D(|A|^{k/2}),D(|A|^{k/2}))}$. From Definition 5, equation (4.1), and $M(v_1 + v_2) \leq M(v_1)M(v_2)$ for any pair (v_1, v_2) in $[0, \delta]^2$ we have

$$M(v) \leq e^{c_k(A,B)|v|}, \qquad \text{for all } v \in \mathbf{R}.$$

Hence, for every n,

$$\|\Upsilon_t^{v'_n}(\psi_0)\|_{k/2} \leq e^{c_k(A,B)K}\|\psi_0\|_{k/2}.$$

For every f in $D(|A|^k)$,

$$|\langle |A|^k f, \Upsilon_t^{v'_n}\psi_0\rangle| \leq \|f\|_{k/2}\|\psi_0\|_{k/2}e^{c_k(A,B)K}.$$

Because of the continuity result (Proposition 7, Corollary 11 and Remark 12), the left hand side tends to $|\langle |A|^k f, \Upsilon_t^u\psi_0\rangle|$ as n tends to infinity. Hence, for every f in $D(|A|^k)$

$$|\langle |A|^k f, \Upsilon_t^u\psi_0\rangle| \leq \|f\|_{k/2}\|\psi_0\|_{k/2}e^{c_k(A,B)K}.$$

As a consequence, $\Upsilon_t^u\psi_0$ belongs to $D((|A|^{k/2})^*) = D(|A|^{k/2})$ and

$$\||A|^{k/2}\Upsilon_t^u\psi_0\| \leq \|\psi_0\|_{k/2}e^{c_k(A,B)K}.$$

□

Remark 25. Assumption (A3.3) implies that (A, B) is 2-mildly coupled. Indeed, if (A, B) is a pair of skew-adjoint operators satisfying Assumption 3, then Assumption (A3.3) implies, see Remark 12, for small $|t|$ that, for every ψ in $D(A)$,

$$\begin{aligned}\||A|e^{-tB}\psi\| = \|Ae^{-tB}\psi\| = \|e^{tB}Ae^{-tB}\psi\| &\leq \|e^{tB}Ae^{-tB}\psi - A\psi\| + \|A\psi\| \\ &\leq (1 + L|t|)\|A\psi\| \leq e^{L|t|}\|A\psi\| = e^{L|t|}\||A|\psi\|\end{aligned}$$

as the map $t \in \mathbf{R} \mapsto e^{tB}Ae^{-tB} \in L(D(A), \mathcal{H})$ is locally Lipschitz with constant L. Thus (A, B) is 2-mildly coupled.

As a consequence of Corollary 11 and Lemma 38 we have the following proposition.

Proposition 28. *Let k be a positive real, let (A, B) satisfy Assumption 3, and let (A, B) be k-mildly coupled. Then for any $s \in [0, k)$, for every ψ_0 in $D(|A|^{s/2})$, the end-point mapping*

$$\begin{aligned} \Upsilon(\psi_0) : BV([0,T], \mathbf{R}) &\to D(|A|^{s/2}) \\ v &\mapsto \Upsilon_T^{\mathfrak{d}v}(\psi_0), \end{aligned}$$

is continuous.

Proof. Let $(v_n)_{n\in\mathbf{N}}$ be a converging sequence in $BV([0,T], \mathbf{R})$ to some v in $BV([0,T], \mathbf{R})$. Then $\Upsilon_T^{\mathfrak{d}v_n}(\psi_0) - \Upsilon_T^{\mathfrak{d}v}(\psi_0)$ is uniformly bounded in $D(|A|^{k/2})$ (by Proposition 27) and converges to 0 in $\mathcal{H}$ (by Proposition 7, Corollary 11 and Remark 12). By Lemma 38 it converges to 0 in $D(|A|^{s/2})$ for $s < k$. $\square$

Remark 26. Under the assumptions of Proposition 28 both Proposition 17 and Corollary 18 extend to $D(|A|^{s/2})$ for $s \in [0, k)$.

5 Bounded control potentials

5.1 Dyson expansion solutions

In the Hilbert setting, if A is maximal dissipative and B stabilizes $D(A)$ Corollary 18 provides an extension of [BMS82, Theorem 3.6] to L^1 controls. This result can be extended to the Banach framework, with A being generator of a strongly continuous semigroup. Below we extend Corollary 18 to bounded control potentials B, for L^1 controls, to the Banach framework without any further assumption in B, providing a proof of Proposition 2.

Troughout this section only, we consider a Banach space $\mathcal{X}$ and we assume that A, acting on $\mathcal{X}$, is the generator of a strongly continuous semigroup with domain $D(A)$ and B is bounded. Then for every u in $\mathbf{R}$, $A + uB$ is also a generator of a strongly continuous semigroup with domain $D(A)$. This can be deduced form an analysis of the Dyson expansion.

Since A generates a strongly continuous semigroup there exist $C_A > 0$ and $\omega \in \mathbf{R}$ such that

$$\|e^{tA}\| \leq C_A e^{\omega t}, \quad \forall t > 0. \tag{5.1}$$

For the equivalent norm

$$N(\psi) = \sup_{t>0} \|e^{t(A-\omega)}\psi\|, \tag{5.2}$$

we have that $A-\omega$ is the generator of a contraction semigroup. If $B \in L(\mathcal{X})$ is bounded for the norm N let $\|B\|_N$ be its norm. Now for every $u \in BV([0,T], [-R, R])$ we consider the family of operators $A - \omega + u(t)B - R\|B\|_N$ which satisfies the assumptions of [Kat53] in the Banach space structure associated with the norm N. So that in this case the results of Section 2.2 are still valid.

It is classical (see [BMS82]) that the input-output mapping Υ admits a unique continuous extension to $L^1(\mathbf{R}, \mathbf{R})$. We consider below the extension to Radon measures.

Definition 6. Let A, with domain $D(A)$, be the generator of a strongly continuous semigroup on $\mathcal{X}$ and let B be bounded on $\mathcal{X}$. We say that $\psi : [0, T] \to \mathcal{X}$ is a mild solution of (2.1) on $[0, T]$ if ψ is bounded on $[0, T]$ and there exists ψ_0 in $\mathcal{X}$ such that, for every t in $[0, T]$,

$$\psi(t) = e^{tA}\psi_0 + \int_{[0,t)} e^{(t-s_1)A} B\psi(s) \mathrm{d}u(s_1). \tag{5.3}$$

If X_1 and X_2 are two metric spaces, we denote by $\mathcal{L}^\infty(X_1, X_2)$ the space of bounded borelian functions from X_1 to X_2. If X_2 is a Banach spaces, then $\mathcal{L}^\infty(X_1, X_2)$ is a Banach space as well, endowed with the sup norm $\|\psi\|_{\mathcal{L}^\infty(X_1,X_2)} = \sup_{t\in X_1} \|\psi(t)\|_{X_2}$.

Remark 27. Definition 6 only makes sense if (5.3) holds for every t in $[0, T]$, since an equality valid almost everywhere only would miss the atoms of u. For this reason, we will consider solutions in $\mathcal{L}^\infty([0, T], \mathcal{X})$ instead of L^∞.

Proposition 29 below states immediate regularity properties of the mild solutions.

Proposition 29. *Let $T > 0$, u in $\mathcal{R}([0, T])$ and ψ be a mild solution of (2.1) on $[0, T]$, associated with ψ_0 in $D(A)$. Then ψ has bounded variation, is left continuous everywhere on $[0, T]$ and $\psi(0) = \psi_0$. Moreover, the discontinuities of ψ (if any) happen on atoms of u, and, for every t in $[0, T]$,*

$$\psi(t + 0) - \psi(t) = \psi(t + 0) - \psi(t - 0) = u(\{t\})B\psi(t) = u(\{t\})B\psi(t - 0).$$

Proof. We have that

$$\psi(t-0) = e^{tA}\psi_0 + \int_{[0,t)} e^{(t-s_1)A}B\psi(s)\mathrm{d}u(s_1) \quad \text{and} \quad \psi(t+0) = e^{tA}\psi_0 + \int_{[0,t]} e^{(t-s_1)A}B\psi(s)\mathrm{d}u(s_1),$$

due to inner and outer regularity of Radon measures. □

Theorem 30. *Let A, with domain $D(A)$, be the generator of a strongly continuous semigroup on $\mathcal{X}$ and let B be bounded on $\mathcal{X}$. Then, for every ψ_0 in $\mathcal{X}$, for every $T > 0$, for any $u \in \mathcal{R}([0, T])$, for every s in $[0, T)$, the Cauchy problem (2.1) with initial condition ψ_0 at time s, admits a unique mild solution $t \mapsto \Xi^u_{t,s}\psi_0$ bounded in $\mathcal{X}$ uniformly on $[s, T]$. That is for every $t \in \Delta_{[0,T]}$*

$$\Xi^u_{t,s}\psi_0 = e^{(t-s)A}\psi_0 + \int_{[s,t)} e^{(t-s_1)A}B\Xi^u_{s_1,s}\psi_0\mathrm{d}u(s_1)$$

$$\sup_{(s,t)\in\Delta_{[0,T]}} \|\Xi^u_{t,s}\psi_0\| < \infty.$$

Moreover

(i) $\Xi^u(s, s) = I_{\mathcal{X}}$,

(ii) $\Xi^u_{t,s} = \Xi^u(t, r)\Xi^u(r, s)$, *for any $s < r < t$,*

(iii) if u has bounded variation on $[0, T]$, for any $\psi_0 \in \mathcal{X}$, $(s, t) \in \Delta_{[0,T]} \mapsto \Xi^u_{t,s}\psi_0$ is strongly continuous in s and t and if $\psi_0 \in D(A)$ then it is strongly right differentiable in t with derivative $(A + u(t + 0)B)\Upsilon^u(t, s)\psi_0$,

(iv) for any $u \in \mathcal{R}([0, T])$, Ξ^u satisfies

$$\|\Xi^u_{t,s}\|_{L(\mathcal{X})} \leq C_A e^{\omega|t-s|+|u|([s,t])C_A\|B\|},$$

(v) for any $r > 0$, $R > 0$, $\psi_0 \in \mathcal{X}$ with $\|\psi_0\| = r > 0$, the set

$$\left\{\Xi^u_{t,s}\psi_0 \mid u \in \mathcal{R}([0, T]), |u|((0, T]) \leq R, (s, t) \in \Delta_{[0,T]}\right\}$$

is relatively compact.

Proof. Let $u \in \mathcal{R}([0,T])$. We first consider the case where ψ_0 belongs to $D(A)$ and we extend the result by density. Then, by Proposition 29, any mild solution of (2.1) taking value ψ_0 at time 0 satisfies

$$\psi(t) = e^{tA}\psi_0^+ + \int_{s\in(0,t)} e^{(t-s)A}B\psi(s)\mathrm{d}u(s),$$

with $\psi_0^+ = \psi_0 + u(\{0\})B\psi_0$. In other words, the restriction to $(0,T]$ of the mild solutions (if any) of (2.1) on $[0,T]$ taking value ψ_0 at time 0 are exactly the fixed points of

$$\begin{array}{llll} F_T^u : & \mathcal{L}^\infty((0,T],\mathcal{X}) & \to & \mathcal{L}^\infty((0,T],\mathcal{X}) \\ & \psi & \mapsto & e^{tA}\psi_0^+ + \int_{s\in(0,t)} e^{(t-s)A}B\psi(s)\mathrm{d}u(s). \end{array}$$

We aim to prove that F_T^u has a unique fixed point in $\mathcal{L}^\infty((0,T],\mathcal{X})$. For this, we will prove that $(F_T^u)^j$ is a contraction from $\mathcal{L}^\infty((0,T],\mathcal{X})$ to itself, for an integer j large enough. We define

$$\begin{array}{llll} G_T^u : & \mathcal{L}^\infty((0,T],\mathcal{X}) & \to & \mathcal{L}^\infty((0,T],\mathcal{X}) \\ & \psi & \mapsto & \int_{s\in(0,t)} e^{(t-s)A}B\psi(s)\mathrm{d}u(s). \end{array}$$

As B is bounded, for every ψ in $\mathcal{L}^\infty((0,T],\mathcal{X})$, for every n in $\mathbf{N}$

$$\begin{aligned} &\| (G_T^u)^n(\psi)\|_{\mathcal{L}^\infty((0,T],\mathcal{X})} \\ &\quad\leq \left\| \int_{0<s_1<s_2<\ldots<s_n<t} e^{(t-s_n)A}Be^{(s_n-s_{n-1})A}\ldots Be^{(s_2-s_1)A}Be^{(s_1-s)A}\psi_0\mathrm{d}u(s_1)\mathrm{d}u(s_2)\ldots\mathrm{d}u(s_n)\right\|_{\mathcal{L}^\infty((0,T],\mathcal{X})} \\ &\quad\leq e^{\omega(t-s)}C_A^{n+1}\|B\|^n\|\psi\|_{\mathcal{L}^\infty((0,T],\mathcal{X})}\int_{0<s_1<s_2<\ldots<s_n<T} \mathrm{d}|u|(s_1)\mathrm{d}|u|(s_2)|\ldots\mathrm{d}|u|(s_n), \end{aligned}$$

and since $(0,T)^n$ contains the disjoint union of $\{0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} < T\}$ over all permutations σ of $\{1,2,\ldots,n\}$

$$\| (G_T^u(\psi))^n \|_{\mathcal{L}^\infty((0,T],\mathcal{X})} \leq e^{\omega T}C_A^{n+1}\|B\|^n\|\psi\|_{\mathcal{L}^\infty((0,T],\mathcal{X})}\frac{|u|((0,T))^n}{n!}.$$

Note that, for $\phi_0^+(t) := e^{tA}\psi_0^+$,

$$(F_T^u)^n(\psi) = \sum_{k=0}^{n-1}(G_T^u)^k(\phi_0^+) + (G_T^u)^n(\psi),$$

converges in $\mathcal{L}^\infty((0,T],\mathcal{X})$.

In particular, there exists n large enough such that $(F_T^u)^n$ is a contraction from $\mathcal{L}^\infty([0,T],\mathcal{X})$ to itself, hence admits a unique fixed point ψ_∞. Since

$$(F_T^u)^n \circ F_T^u(\psi_\infty) = (F_T^u)^{n+1}(\psi_\infty) = F_T^u \circ (F_T^u)^n(\psi_\infty) = F_T^u(\psi_\infty),$$

we have that $F_T^u(\psi_\infty)$ is also a fixed point of $(F_T^u)^n$, hence $F_T^u(\psi_\infty) = \psi_\infty$, or, in other words, ψ_∞ is the restriction to $(0,T]$ of the unique mild solution $t \mapsto \Xi_{t,0}^u(\psi_0)$ of (2.1) on $[0,T]$ taking value ψ_0 at time 0. Conversely, noticing that the restriction to $(0,T]$ of any mild solution of (2.1), with initial condition ψ_0, is a fixed point of F_T^u provides the uniqueness of the mild solution of (2.1).

Setting the initial time at s instead of 0, the linear map $\Xi_{t,0}^u : \mathcal{X} \to \mathcal{X}$ extends to $t \geq s$, by $\Xi_{t,s}^u : \mathcal{X} \to \mathcal{X}$ such that

$$\Xi_{t,s}^u\psi_0 = e^{(t-s)A}\psi_0 + \int_{r\in[s,t)} e^{(t-r)A}B\Xi_{r,s}^u\psi_0\mathrm{d}u(r).$$

The core idea of the Dyson expansion is to express $\Xi^u_{t,s}$ as the sum of a series. Precisely, we define for every $n \in \mathbf{N}$, the linear operator

$$W^u_{(n)}(t,s) : \psi_0^+ \in \mathcal{X} \mapsto \int_{s<s_1<s_2<\ldots<s_n<t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \ldots B e^{(s_2-s_1)A} B e^{(s_1-s)A} \psi_0^+ \mathrm{d}u(s_1)\mathrm{d}u(s_2)\ldots\mathrm{d}u(s_n).$$

Notice that

$$W^u_{(0)}(t,s)\psi_0^+ = e^{(t-s)A}\psi_0^+, \quad \text{and} \quad W^u_{(n+1)}(t,s)\psi_0^+ = \int_{(s,t)} e^{(t-\tau)A} B W^u_{(n)}(\tau,s)\psi_0^+ \mathrm{d}u(\tau).$$

Using the very same computation used above to prove that G^u_T is a contraction, we get

$$\|W^u_{(n)}(t,s)\psi_0^+\| \leq e^{\omega(t-s)} C_A^{n+1} \|B\|^n \|\psi_0^+\| \frac{|u|((s,t))^n}{n!},$$

which proves by uniqueness that

$$\Xi^u_{t,s} = \sum_{n=0}^{\infty} W^u_{(n)}(t,s) \circ (1 + u(\{s\})B), \tag{5.4}$$

converges in norm in the set $L(\mathcal{X})$ of the bounded operators of $\mathcal{X}$. This also provides

$$\|\Xi^u_{t,s}\|_{L(\mathcal{X})} \leq C_A(1 + |u|([s,t)\|B\|) e^{\omega|t-s| + |u|([s,t))C_A\|B\|},$$

and the fact that $\Xi^u_{t,s}$ is a bounded map, which admits an extension to $\mathcal{X}$ by density of $D(A)$. By abuse of notation, we still denote this extension with $\Xi^u_{t,s}$.

Then we have that

$$\begin{aligned}
\Xi^u_{t,s}\psi_0 &= e^{(t-s)A}\psi_0 + \int_{[s,t)} e^{(t-s_1)A} B \Xi^u_{s_1,s}\psi_0 \mathrm{d}u(s_1) \\
&= e^{(t-s)A}\psi_0 + \int_{[s,r)} e^{(t-s_1)A} B \Xi^u_{s_1,s}\psi_0 \mathrm{d}u(s_1) + \int_{[r,t)} e^{(t-s_1)A} B \Xi^u_{s_1,s}\psi_0 \mathrm{d}u(s_1) \\
&= e^{(t-r)A}\Xi^u_{r,s}\psi_0 + \int_{[r,t)} e^{(t-s_1)A} B \Xi^u_{s_1,s}\psi_0 \mathrm{d}u(s_1) \\
&= \Xi^u_{t,r}\Xi^u_{r,s}\psi_0
\end{aligned}$$

where we used the uniqueness in the last identity.

The differentiability properties in the bounded variation case are due to [Kat53, Theorem 1]. Indeed recall that $A - \omega$ is the generator of a contraction semigroup and since B in $L(\mathcal{X})$ then B is bounded for the norm N defined in (5.2). So that $A - \omega + u(t)B - R\|B\|_N$ for any $R > |u|_\infty$ satisfies the assumptions of [Kat53, Theorem 1] in the Banach space $\mathcal{X}$ with norm N.

We now consider the compactness property in the last statement. Without loss of generality by linearity and up to scaling B, we can assume $r = R = 1$. Let us prove that, for $\|\psi_0\| = 1$,

$$\left\{\Xi^u_{t,s}\psi_0 \mid u \in \mathcal{R}([0,T]), |u|([0,T]) \leq 1, (s,t) \in \Delta_{[0,T]}\right\}$$

is totally bounded for the topology of $\mathcal{X}$. Then its closure will be totally bounded and complete and thus compact.

Let us consider a radius $\epsilon > 0$. In place of $\Xi^u_{t,s}\psi_0$, due to its norm convergence we can consider one of the truncated series in (5.4), namely

$$\sum_{n=0}^{n_\epsilon} W^u_{(n)}(t,s) \circ (1 + u(\{s\})B),$$

for some $n_\epsilon \in \mathbf{N}$ such that

$$\|\Xi^u_{t,s} - \sum_{n=0}^{n_\epsilon} W^u_{(n)}(t,s) \circ (1 + u(\{s\})B)\| \leq (1 + |u|([s,t)\|B\|) \sum_{n=n_\epsilon+1}^{\infty} e^{\omega T} C_A^{n+1} \|B\|^n \|\psi_0\| \frac{1}{n!} \leq \epsilon.$$

Since we consider a finite number of $W^{\cdot}_{(n)}(\cdot,\cdot) \circ (1 + u(\{s\})B)$, namely n_ϵ of them, it is then enough to prove that

$$\mathcal{W}^T_n := \left\{ W^u_n(t,s) \circ (1 + u(\{s\})B)\psi_0 \mid u \in \mathcal{R}([0,T]), |u|([0,T]) \leq 1, (s,t) \in \Delta_{[0,T]} \right\},$$

is totally bounded for the $\mathcal{X}$ topology for any integer n. This will be done by iteration on $n \in \mathbf{N} \cup \{0\}$:

- For $n = 0$, $W^u_0(t,s) \circ (1 + u(\{s\})B)\psi_0 = e^{(t-s)A} \circ (1 + u(\{s\})B)\psi_0$ and since $\Delta_{[0,T]}$ is compact and $|u(\{s\})| \leq 1$, the strong continuity in time of the semigroup provides the compactness of $\mathcal{W}^T_0$.
- For any integer n, we now assume $\mathcal{W}^T_n$ is totally bounded. The map

$$(\tau, t, \psi) \in \Delta_{[0,T]} \times \mathcal{X} \mapsto e^{(t-\tau)A} B\psi \in \mathcal{X},$$

is continuous. So

$$\mathcal{Z}^T_n := \left\{ e^{(t-\tau)A} B W^u_n(\tau,s) \circ (1 + u(\{s\})B)\psi_0 \mid u \in \mathcal{R}([0,T]), |u|([0,T]) \leq 1, (s,\tau) \in \Delta_{[0,T]}, (\tau,t) \in \Delta_{[0,T]} \right\}$$

is totally bounded.

For any $\delta > 0$, there exist $\psi_1, \ldots, \psi_{N_\delta}$ in $\mathcal{X}$ such that

$$\mathcal{Z}^T_n \subset \cup_{j=1}^{N_\delta} B_{\mathcal{X}}(\psi_j, \delta).$$

Let $\phi_1, \ldots, \phi_{N_\delta}$ be a partition of the unity in $\mathcal{Z}^T_n$ such that supp $\phi_j \subset \overline{B_{\mathcal{X}}(\psi_j, 2\delta)}$ and $\pi : \psi \in \mathcal{X} \mapsto \sum_{j=1}^{N_\delta} \psi_j \phi_j(x)$.

Define $p^u_n(t,\tau,s) := \pi(e^{(t-\tau)A} B W^u_n(\tau,s) \circ (1 + u(\{s\})B)\psi_0)$ and $\phi^u_{n,j}(t,\tau,s) := \phi_j(e^{(t-\tau)A} B W^u_n(\tau,s) \circ (1 + u(\{s\})B)\psi_0)$, then

$$p^u_n(t,\tau,s) = \sum_{j=1}^{N_\delta} \psi_j \phi^u_{n,j}(t,\tau,s)$$

and

$$\|e^{(t-\tau)A} B W^u_n(\tau,s) \circ (1 + u(\{s\})B)\psi_0 - p^u_n(t,\tau,s)\| \leq 2\delta.$$

Thus $\mathcal{W}^T_{n+1}$ is totally bounded if

$$\mathcal{P}^T_n := \left\{ \int_{(s,t)} p^u_n(t,\tau,s) u(\tau) \mathrm{d}\tau, u \in \mathcal{R}([0,T]), |u|([0,T]) \leq 1, (s,\tau) \in \Delta_{[0,T]}, (\tau,t) \in \Delta_{[0,T]} \right\}$$

is totally bounded. Since for $u \in \mathcal{R}([0,T])$, $|u|((0,T]) \leq 1$, $(s,\tau) \in \Delta_{[0,T]}$ and $(\tau,t) \in \Delta_{[0,T]}$

$$\int_{(s,t)} p^u_n(t,\tau,s) \mathrm{d}u(\tau) = \sum_{j=1}^{N_\delta} \psi_j \int_{(s,t)} \phi^u_{n,j}(t,\tau,s) \mathrm{d}u(\tau)$$

and

$$\left| \int_{(s,t)} \phi^u_{n,j}(t,\tau,s) \mathrm{d}u(\tau) \right| \leq |u|((0,T]) \leq 1$$

this implies $\mathcal{P}^T_n$ is relatively compact (and thus totally bounded).

This concludes the iteration. We thus have the relative compactness of

$$\left\{\Xi^u_{t,s}\psi_0 \mid u \in \mathcal{R}([0,T]), |u|([0,T]) \leq 1, (s,t) \in \Delta_{[0,T]}\right\},$$

concluding the proof. ☐

Corollary 31. *Let A be the generator of a strongly continuous semigroup on $\mathcal{X}$, let B be bounded and denote by Ξ the propagator defined in Theorem 30. Then for every ψ_0 in $\mathcal{X}$, the set*

$$\mathcal{A}tt^{\Xi}_{\mathcal{R}}(\psi_0) := \bigcup_{T\geq 0} \bigcup_{u\in\mathcal{R}([0,T])} \{\Xi^u_{t,0}\psi_0 | t \in [0,T]\}$$

is contained in a countable union of compact subsets of $\mathcal{X}$.

Proof. The proof is a consequence of Theorem 30 and follows the idea of the proof of Corollary 18. ☐

We are now ready to prove Proposition 2.

Proof of Proposition 2. As already mentioned, the well-posedness result is classical (see [BMS82] for instance), while the property of the attainable set with L^1 controls follows from Corollary 31. ☐

Remark 28. If $\mathcal{X}$ is an Hilbert space $\mathcal{H}$, A is skew-adjoint, B is bounded in $D(|A|^{k/2})$, then $D(|A|^{k/2})$ can be considered in stead of $\mathcal{X}$ in the whole analysis of the present section. This leads to results similar to the ones presented in Section 4 on the mild coupling, but in a simpler way.

5.2 On the notion of solution in the Radon framework

Theorem 30 does not deal with the continuity with respect to the control u. With a Dirac measure δ_{t_0}, $t_0 \in (0,T]$, it turns out that the solution built here is

$$\Xi^u_{t,s}\psi_0 = e^{(t-s)A}\psi_0 + e^{(t-t_0)A}Be^{(t_0-s)A}\psi_0\mathbb{I}_{[s,t)}(t_0). \tag{5.5}$$

This does not coincide with the generalized propagator in Definition 4 even if the framework is similar, for instance, as in Remark 18, when $A = 0$ as the latter is

$$\Upsilon^u_{t,s}\psi_0 = e^{B\mathbb{I}_{(s,t]}(t_0)}\psi_0.$$

Both expansions coincide only up to the first order term in the control. This discrepancy is due to the lack of continuity of the cumulative function of the control.

If we restrict the analysis to controls with continuous cumulative functions and set the topology to the one of the total variation, the continuity is restored and both constructions thus coincide.

As a consequence the propagator in Theorem 30 is not continuous in u and it is not the sequential extension of the corresponding propagator for, say, controls with continuous cumulative functions when the notion of convergence is the one we choose for $\mathcal{R}([0,T])$. This also implies that the accumulations of the compact set

$$\left\{\Xi^u_{t,s}\psi_0 \mid u \in \mathcal{R}([0,T]), t \mapsto u((0,t]) \text{ is continuous }, |u|((0,T]) \leq 1, (s,t) \in \Delta_{[0,T]}\right\},$$

are not necessarily given by values of the propagator in Theorem 30 and thus actual solutions but, instead, values of the propagator in the sense of Definition 4.

5.3 Noninvariance of the domain

In this section, we consider the invariance of the domain of A, in the framework of Theorem 30, by Ξ^u when u is in $L^1([0,T],\mathbf{R})$. Notice that if u is in $L^1([0,T],\mathbf{R})$, Υ^u and Ξ^u coincide whenever Assumption 2 is fulfilled and, hence, the invariance of the domain follows from Theorem 5. The question is whether this remains true when B is bounded but the corresponding C^0-semigroup does not preserve $D(A)$. The answer is negative and we provide a counter-example.

Let $\mathcal{X} = L^2(\mathbf{R})$, $A = \partial_x$ with $D(A) = H^1(\mathbf{R})$ and $B = \mathrm{i}w$ for some bounded measurable function w. This provides a controlled transport equation and the corresponding solution of (1.1) with $u \in L^1(\mathbf{R})$ is given by

$$\Xi_t^u(\psi_0)(x) = e^{\frac{\mathrm{i}}{2}\int_{-t-x}^{t-x} u(\frac{t-x+\tau}{2})w(\frac{t+x-\tau}{2})\,\mathrm{d}\tau}\psi_0(t+x)$$

which rewrites as

$$\Xi_t^u(\psi_0)(x) = e^{\mathrm{i}\int_x^{t+x} u(t-s)w(s)\,\mathrm{d}s}\psi_0(t+x).$$

Set $w = \mathbb{I}_{[0,+\infty)}$, then for $t \geq 0$ and $x \geq 0$

$$\Xi_t^u(\psi_0)(x) = e^{\mathrm{i}\int_{[-x,t-x]} u(s)\,\mathrm{d}s}\psi_0(t+x).$$

For fixed time t, the function $x \mapsto e^{\mathrm{i}\int_{[-x,t-x]} u(s)\,\mathrm{d}s}$ is absolutely continuous and the distributional derivative of $x \mapsto \Xi_t^u(\psi_0)(x)$ is given by

$$\Xi_t^u(\psi_0)(x) = e^{\mathrm{i}\int_{[-x,t-x]} u(s)\,\mathrm{d}s}\left(\psi_0'(t+x) + \mathrm{i}(u(-x)-u(t-x))\psi_0(t+x)\right),$$

for $t > 0$ and $x > 0$.

If ψ_0 is in $H^1(\mathbf{R})$ then $\Xi_t^u(\psi_0)$ is in $H^1(\mathbf{R})$ if and only if

$$v : x \mapsto (u(-x)-u(t-x))\psi_0(t+x)$$

is in $L^2(\mathbf{R})$.

Set $u : t \mapsto |1-t|^{-1/2}$, which is integrable but not square integrable, and ψ_0 a smooth compactly supported function equal to 1 in $[1-\varepsilon, 1+\varepsilon]$, for some $\varepsilon \in (0,1/2)$. Consider $t \in [1-\varepsilon/2, 1+\varepsilon/2]$, then $x \mapsto \psi_0(t+x)$ is equal to 1 on $[1-t-\varepsilon, 1-t+\varepsilon] \subset [-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon] \subset [-3/4, 3/4]$. Hence $-1 \notin [1-t-\varepsilon, 1-t+\varepsilon]$. While $[-\varepsilon/2, \varepsilon/2] \subset [1-t-\varepsilon, 1-t+\varepsilon]$ and $x = t-1 \in [-\varepsilon/2, \varepsilon/2]$. This implies that v is not square integrable on $[1-t-\varepsilon, 1-t+\varepsilon]$ for any $t \in [1-\varepsilon/2, 1+\varepsilon/2]$.

6 Examples

Most of the examples of bilinear control systems (2.1) encountered in the literature, also the ones not related to quantum control, deal with bounded control operator B. Proposition 2 applies and allows, for instance, to complete the studies of the rod equation with clamped ends made in [BMS82, Section 6, Example 4] and [Bea08]. In the following, we focus on examples arising in quantum control.

6.1 Quantum systems with smooth potentials on compact manifolds

The following example motivates the present analysis because of its physical relevance. We consider Ω a compact Riemannian manifold endowed with the associated Laplace-Beltrami operator Δ and the associated measure μ. For r a positive real, $D(|\Delta|^{\frac{r}{2}}) = H^r(\Omega,\mathbf{C})$. Since Ω is compact, for $r > r'$, $D(|\Delta|^{\frac{r}{2}}) \subset D(|\Delta|^{\frac{r'}{2}})$ is a compact embedding.

Let $k \in \mathbf{N}$, let $V, W : \Omega \to \mathbf{R}$ be two functions of class $C^{2(k-1)}$, and consider bilinear quantum system

$$\mathrm{i}\frac{\partial \psi}{\partial t} = \Delta\psi + V\psi + u(t)W\psi. \tag{6.1}$$

Following the notations of Section 2, $\mathcal{H} = L^2(\Omega, \mathbf{C})$ is endowed with the Hilbert product $\langle f, g\rangle = \int_\Omega \bar{f} g \mathrm{d}\mu$, $A = -\mathrm{i}(\Delta + V)$, and $B = -\mathrm{i}W$. As V is continuous and so bounded, A has a spectral gap. Up to substracting a sufficiently large constant, we can assume that A is positive and invertible.

For a positive real r with $r \leq 2k$, $D(|A|^{\frac{r}{2}}) = H^r(\Omega, \mathbf{C})$. Since B is bounded from $D(|A|^{\frac{s}{2}})$ to $D(|A|^{\frac{s}{2}})$, for s a postive real with $s \leq 2(k-1)$, $(A, B, \mathbf{R})$ satisfies Assumption 1 and (A, B) is s-mildly coupled by Proposition 20.

In particular, the two notions of propagators Υ and Ξ defined in Proposition 15 and Theorem 30 respectively can be used and we have the following statement.

Proposition 32. *For every $T > 0$, for every ψ_0 in $H^{2(k-1)}(\Omega, \mathbf{C})$, the sets*

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0,T])} \{\alpha \Upsilon^u_t \psi_0 \mid t \in [0, T]\},$$

and

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0,T])} \{\alpha \Xi^u_t \psi_0 \mid t \in [0, T]\}$$

are contained in countable unions of compact subsets of $H^{2(k-1)}(\Omega, \mathbf{C})$ and, in particular, they have dense complement in $H^{2(k-1)}$.

For any $\varepsilon \in (0, 1)$, if ψ_0 in $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$, the set

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in BV([0,T], \mathbf{R})} \{\alpha \Upsilon^u_t \psi_0 \mid t \in [0, T]\}$$

is contained in a countable union of compact subsets of $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$ and, in particular, it has dense complement in $H^{2(k-\varepsilon)}(\Omega, \mathbf{C})$.

Proof. The first statement is an adaptation of Proposition 17 and Corollary 18, see Remark 26. The second statement follows from Corollary 31. The last statement is a consequence of Corollary 26. ☐

Notice that from the compactness of the Sobolev embeddings and the conservation of the regularity we can deduce a result weaker than Proposition 32 such as the fact

$$\bigcup_{\alpha \geq 0} \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0,T])} \{\alpha \Upsilon^u_t \psi_0 \mid t \in [0, T]\}$$

is contained in a countable union of totally bounded sets of $H^{2(k-1)-\delta}$ for any $\delta \in (0, 1)$ whenever ψ_0 in $H^{2(k-1)}$.

6.2 Potential well with dipolar interaction

In this example, $\Omega = (0, \pi)$ is endowed with the standard Lebesgue measure, V is the constant zero function and W is some function of class C^k, for some integer k. This academic example is a simplification of the harmonic oscillator, presented in Section 6.3 below, in the sense that Ω is bounded. It has been thoroughly studied by K. Beauchard in [Bea05, BL10]. These works give the first (and, at this time, almost the only one) satisfying description of the reachable set with L^2

controls from the first eigenvector for systems of the type of (1.4). Using Lyapunov techniques, V. Nersesyan [Ner10] gave practical algorithms for approximate controllability.

Equation (1.4) writes

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} - u(t)W(x)\psi \tag{6.2}$$

with boundary conditions $\psi(0) = \psi(\pi) = 0$.

The linear operators $A = \frac{\mathrm{i}}{2}\Delta$ defined on $D(A) = (H^2 \cap H_0^1)((0,\pi), \mathbf{C})$ and $B : \psi \mapsto \mathrm{i}W\psi$ are skew symmetric in the Hilbert space $\mathcal{H} = L^2(\Omega, \mathbf{C})$ endowed with the hermitian product $L^2(\Omega, \mathbf{C})$,

$$\langle f, g\rangle = \int_0^\pi \overline{f(x)} g(x) \mathrm{d}x.$$

Define, for every k in $\mathbf{N}$,

$$\phi_k : x \mapsto \sqrt{\frac{2}{\pi}} \sin(kx),$$

the family $\Phi = (\phi_k)_{k\in\mathbf{N}}$ is an orthonormal basis of $\mathcal{H}$ made of eigenvectors of A.

The triple $(A, B, \mathbf{R})$ satisfies Assumption 1.

Classical results of interpolation [LM68, Chapter 1] allow to find the domain of fractional derivative operators. In particular, for any k in $\mathbf{N}$ and $0 \leq s < 1$, we get following ([NOS16]):

$$\begin{array}{lcll}
D(|A|^k) & = & \{\psi \in H^{2k} | \psi^{[2l]}(0) = \psi^{[2l]}(\pi) = 0, l = 0, \ldots, k-1\} & \text{for } k \in \mathbf{N} \\
D(|A|^{k+s}) & = & D(|A|^k) \cap H^{2s} & \text{for } s < 1/4 \\
D(|A|^{k+\frac{1}{4}}) & = & \{\psi \in D(|A|^k) | |A|^k \psi \in H_{00}^{\frac{1}{2}}\} & \\
D(|A|^{k+s}) & = & \{\psi \in D(|A|^k) | |A|^k \psi \in H_0^{2s}\} & \text{for } 1/4 < s < 1/2 \\
D(|A|^{k+s}) & = & \{\psi \in D(|A|^k) | |A|^k \psi \in H^{2s} \cap H_0^1\} & \text{for } 1/2 \leq s \leq 1
\end{array}$$

where

$$H_{00}^{\frac{1}{2}} = \left\{ \psi \in H^{\frac{1}{2}} \Big| \int_0^\pi \psi^2(x) \frac{\mathrm{d}x}{\sin(x)} < +\infty \right\}$$

is the Lions-Magenes space.

Lemma 33. *Let p in $\mathbf{N} \cup \{0\}$, $W : [0,\pi] \to \mathbf{R}$ be $C^3 \cap C^{2p+1}$. If $p > 0$, assume moreover that that $W^{(2l+1)}(0) = W^{(2l+1)}(\pi) = 0$ for $l = 0, \ldots, p-1$. Then B is bounded from $D(|A|^a)$ to $D(|A|^a)$ for every $a < p + 1 + \frac{1}{4}$.*

Proof. Since W is C^{2p+1}, B leaves invariant H^s for every $s \leq 2p+1$. If a is an integer less than or equal to $p+1$, the result follows from the Leibniz rule, using the vanishing of the derivatives of odd orders less than $2p$ (if any) of W on the boundary of $[0,\pi]$. The result for $a - \lfloor a \rfloor < 1/4$ follows directly from the equalities above with no additional boundary conditions to check. □

Theorem 3.6 in [BMS82] by Ball, Marsden and Slemrod implies (see [Tur00]) that equation (1.4) is not controllable in (the Hilbert unit sphere of) $L^2(\Omega)$ when $\psi \mapsto W\psi$ is bounded in $L^2(\Omega)$. Moreover, in the case in which Ω is a domain of $\mathbf{R}^n$ and $W : \Omega \to \mathbf{R}$ is C^2, if the control u belongs to $L^p([0,+\infty), \mathbf{R})$ with $p > 1$, then equation (1.4) is neither controllable in the Hilbert sphere $\mathbf{S}$ of $L^2(\Omega)$ nor in the natural functional space where the problem is formulated, namely the intersection of $\mathbf{S}$ with the Sobolev spaces $H^2(\Omega)$ and $H_0^1(\Omega)$.

The fact that the present system is not more than 5/2-mildly coupled is the purpose of the following lemmas.

Lemma 34. *Let $k \in \mathbf{N} \cup \{0\}$. Let $F : [0,\pi] \to \mathbf{R}$ be of class C^{2k+3} with $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$ and, if $k \neq 0$, $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$ for $j = 0, \ldots, k-1$. Then $F\phi_1$ is not in $D(|A|^a)$ if $a \geq k + \frac{5}{4}$.*

Proof. Consider, for any integer n, the following quantity

$$I_n(F) := \frac{\pi}{2}\langle F\phi_1, \phi_n\rangle = \int_0^\pi F(x)\sin(x)\sin(nx)\,\mathrm{d}x.$$

Then we have that $I_n(F) = \frac{1}{2}(J_{n-1}(F) - J_{n+1}(F))$ with

$$\begin{aligned} J_\ell(F) &:= \int_0^\pi F(x)\cos(\ell x)\,\mathrm{d}x = -\frac{1}{\ell}\int_0^\pi F'(x)\sin(\ell x)\,\mathrm{d}x \\ &= \frac{1}{\ell^2}\Big((-1)^\ell F'(\pi) - F'(0)\Big) - \frac{1}{\ell^2}J_\ell(F''). \end{aligned}$$

Now assume that $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$ for $j = 0, \dots, k-1$, hence

$$J_\ell(F) = \frac{1}{\ell^{2k+2}}\Big((-1)^\ell F^{(2k+1)}(\pi) - F^{(2k+1)}(0)\Big) - \frac{1}{\ell^{2k+2}}J_\ell(F^{(2k+2)}).$$

It follows that

$$\begin{aligned} I_n(F) &= \frac{1}{2}\left(\frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}}\right)\Big(-(-1)^n F^{(2k+1)}(\pi) - F^{(2k+1)}(0)\Big) \\ &\quad - \frac{1}{2}\frac{1}{(n-1)^{2k+2}}J_{n-1}(F^{(2k+2)}) + \frac{1}{2}\frac{1}{(n+1)^{2k+2}}J_{n+1}(F^{(2k+2)}) \\ &= \frac{1}{2}\left(\frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}}\right)\Big(-(-1)^n F^{(2k+1)}(\pi) - F^{(2k+1)}(0)\Big) \\ &\quad + \frac{1}{2}\frac{1}{(n-1)^{2k+3}}\int_0^\pi F^{(2k+3)}(x)\sin((n-1)x)\,\mathrm{d}x - \frac{1}{2}\frac{1}{(n+1)^{2k+3}}\int_0^\pi F^{(2k+3)}(x)\sin((n+1)x)\,\mathrm{d}x \end{aligned}$$

As

$$\frac{1}{(n-1)^{2k+2}} - \frac{1}{(n+1)^{2k+2}} = \frac{(n+1)^{2k+2}}{(n^2-1)^{2k+2}}\left(1 - \left(\frac{n-1}{n+1}\right)^{2k+2}\right) \underset{n\to\infty}{\sim} \frac{4k+4}{n^{2k+3}}$$

If $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$, then either $F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \neq 0$ or $F^{(2k+1)}(\pi) + F^{(2k+1)}(0) \neq 0$, and, due to Riemann–Lebesgue Lemma,

- if $F^{(2k+1)}(\pi) + F^{(2k+1)}(0) \neq 0$ then

  $$I_{2n}(F) \underset{n\to\infty}{\sim} -\frac{2k+2}{(2n)^{2k+3}}\left(F^{(2k+1)}(\pi) + F^{(2k+1)}(0)\right),$$

  and hence, $(n^{2a}I_n(F))_{n\in\mathbf{N}}$ is not square integrable if $2a - 2k - 3 \geq -\frac{1}{2}$ and consequently $F\phi_1$ is not in $D(|A|^a)$ if $a \geq k + \frac{5}{4}$

- if $F^{(2k+1)}(\pi) - F^{(2k+1)}(0) \neq 0$ then

  $$I_{2n+1}(F) \underset{n\to\infty}{\sim} \frac{2k+2}{(2n+1)^{2k+3}}\left(F^{(2k+1)}(\pi) - F^{(2k+1)}(0)\right)$$

  and similarly $F\phi_1$ is not in $D(|A|^a)$ if $a \geq k + \frac{5}{4}$. □

Lemma 35. *Let $k \in \mathbf{N}\cup\{0\}$. Let $W : [0,\pi] \to \mathbf{R}$ be of class C^{2k+3} with $|W^{(2k+1)}(\pi)|+|W^{(2k+1)}(0)| \neq 0$ and, if $k \neq 0$, $W^{(2j+1)}(0) = W^{(2j+1)}(\pi) = 0$ for $j = 0, \dots, k-1$. Then for every a in $(0, +\infty)$,*

$$e^{\mathrm{i}W}\phi_1 \in D(|A|^a) \Leftrightarrow a < \frac{5}{4} + k.$$

Proof. Set $F = e^{\mathrm{i}W}$ and recall Faà di Bruno formula

$$(e^{\mathrm{i}W})^{(n)}(x) = \sum \frac{n!}{m_1!\,1!^{m_1}\,m_2!\,2!^{m_2}\,\cdots\,m_n!\,n!^{m_n}} e^{\mathrm{i}W(x)} \prod_{j=1}^{n} \left((\mathrm{i}W)^{(j)}(x) \right)^{m_j},$$

where the sums is over the n-uplets $(m_1, \ldots, m_n)$ in $\mathbf{N} \cup \{0\}$ such that: $1m_1 + 2m_2 + 3m_3 + \cdots + nm_n = n$.

If n is odd and $(m_1, \ldots, m_n)$ is an n-uplets of $\mathbf{N} \cup \{0\}$ such that $1m_1 + 2m_2 + 3m_3 + \cdots + nm_n = n$ there exists ℓ such that $2\ell + 1 \leq n$ and $m_{2\ell+1} \neq 0$. It follows that $F : [0, \pi] \to \mathbf{R}$ is of class C^{2k+3} with $F^{(2j+1)}(0) = F^{(2j+1)}(\pi) = 0$ for $j = 0, \ldots, k-1$ and $|F^{(2k+1)}(\pi)| + |F^{(2k+1)}(0)| \neq 0$.

Then the conclusion follows from Lemma 33 and Lemma 34. $\square$

We sum up our results in the following

Proposition 36. *Let $k \in \mathbf{N} \cup \{0\}$. Let $W : [0, \pi] \to \mathbf{R}$ of class C^{2k+3} with $|W^{(2k+1)}(\pi)| + |W^{(2k+1)}(0)| \neq 0$ and, if $k \neq 0$, $W^{(2j+1)}(0) = W^{(2j+1)}(\pi) = 0$ for $j = 0, \ldots, k-1$. Then*

$$\mathcal{A}tt_{\mathcal{R}}(\phi_1) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0,T])} \{\Upsilon^u_{t,0}\phi_1 | 0 \leq t \leq T\} \subset \bigcap_{s < \frac{5}{4} + k} D(|A|^s),$$

$$\mathcal{A}tt^{\Xi}_{\mathcal{R}}(\phi_1) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{R}([0,T])} \{\Xi^u_{t,0}\phi_1 | 0 \leq t \leq T\} \subset \bigcap_{s < \frac{5}{4} + k} D(|A|^s),$$

and both attainable sets are contained in a countable union of relatively compact subsets of $D(|A|^s)$, for any $s < \frac{5}{4} + k$.

Moreover, we have

$$\mathcal{A}tt_{\mathcal{R}}(\phi_1) \not\subset D(|A|^{\frac{5}{4}+k}) \quad \text{and} \quad \mathcal{A}tt^{\Xi}_{\mathcal{R}}(\phi_1) \not\subset D(|A|^{\frac{5}{4}+k}).$$

Recall that Υ is defined in Proposition 15 and Ξ in Theorem 30.

Proof. From Lemma 34, B is bounded from $D(|A|^a)$ to $D(|A|^a)$ for every $a < p + \frac{5}{4}$ and hence (A, B) is a-mildly coupled, for every $a < p + \frac{5}{4}$, by Proposition 20. Then Proposition 27 provides the first statement. While following Remark 28, Theorem 30 provides the second statement.

The relative compactnesses follow from Proposition 28 (similarly to Proposition 17) and Theorem 30, respectively.

From (3.2), with $u = \pi\delta_{t_0}$ for some $t_0 > 0$ and Lemma 35 we deduce the first noninclusion statement. From (5.5) and Lemma 34, we deduce the last assertion. $\square$

Remark 29. Notice that [BL10, Theorem 2] states the exact controllability of (6.2) in $D(|A|^{\frac{5}{2}})$ with H^1_0 controls and $W : x \mapsto x^2$. While Proposition 25 implies nonexact controllability of (6.2) in $D(|A|^s)$, $s < \frac{9}{4}$, with BV controls for example with $W : x \mapsto x^2$. Whether this 1/4 discrepancy is optimal is still an open question.

Similarly [BL10, Theorem 1] states the exact controllability of (6.2) in $D(|A|^{\frac{3}{2}})$ with L^2 controls and $W : x \mapsto x$. While Proposition 28 states the nonexact controllability of (6.2) in $D(|A|^s)$, $s < \frac{5}{4}$, with Radon controls and $W : x \mapsto x^2$. But this time, from Proposition 28, we deduce that the 1/4 discrepancy is optimal.

From [BCCS12], we know that $\{(k, k+1) | k \in \mathbf{N}\}$ is a nondegenerate chain of connectedness for $(A + \eta B, B)$ for almost every real η. Hence Proposition 45 guarantees the approximate controllability of the system (6.2) from ϕ_1 in $D(|A + \eta B|^r) = D(|A|^r)$, for $\frac{3}{2} < r < \frac{5}{4} + 1$. The global exact controllability in $D(|A|^{\frac{3}{2}})$ (inside the unit sphere) with explicit controls follows from Proposition 45, in order to reach a neighborhood of the target in $D(|A|^r)$, for $\frac{3}{2} < r < \frac{5}{4} + 1$ (see for instance [BCC12]). It is then enough to concatenate the dynamics with L^2 controls given by [BL10] for exact local controllability. This explicit construction provides estimates on control time and norms, see [Duc17].

6.3 Quantum harmonic oscillator

In this section, we present an example of s-mildly coupled system, for any $s > 0$, with an unbounded control potential, in contrast with the examples in the previous sections.

The quantum harmonic oscillator with angular frequency ω describes the oscilations of a particle of mass m subject to the potential $V(x) = \frac{1}{2}m\omega x^2$. The corresponding uncontrolled Schrödinger equation is

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(x,t) + \frac{1}{2}m\omega x^2\psi(x,t).$$

With a suitable choice of units, it reads

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi(x,t) + \frac{1}{2}x^2\psi(x,t).$$

The operator $A = \frac{\mathrm{i}}{2}\Delta - \frac{x^2}{2}$ is self-adjoint on $L^2(\mathbf{R},\mathbf{C})$ and it has a pure discrete spectrum. The k^{th} eigenvalue (corresponding to the k^{th} energy level) is equal to $\frac{2k+1}{2}\mathrm{i}$ and it is associated with the eigenstate

$$\phi_k : x \mapsto \frac{1}{\sqrt{2^k k!\sqrt{\pi}}}\exp\left(-\frac{x^2}{2}\right)H_k(x),$$

where H_k is the k^{th} Hermite polynomial, namely $H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}\left(e^{-x^2}\right)$.

When considering the classical dipolar interaction, the control potential W takes the form $W(x) = x$ for every x in $\mathbf{R}$. It is well known (see [MR04] and references therein) that the resulting control system (1.4) is not controllable in any reasonable sense. Indeed the system splits in two uncoupled subsystems. The first one is a finite dimensional classical harmonic oscillator which is controllable. The second one is a free (that is, without control) quantum harmonic oscillator, whose evolution does not depend on the control and is therefore not controllable.

In [BCC13, Section IV], we show that $(\mathrm{i}(-\Delta + V), \mathrm{i}W)$ is s-mildly coupled for every $s > 0$. The proof given in [MR04, ILT06] (and especially the decomposition of the system in two decoupled systems) does not require more to the control than to be the derivative of a derivable function. Using the continuity in Proposition 15, the noncontrollability result can be extended to Radon measures.

Proposition 37. *The system* (1.4) *with* $\Omega = \mathbf{R}$, $V : x \mapsto x^2$ *and* $W : x \mapsto x$ *is not approximately controllable by means of Radon measures.*

Although this example is not approximately controllable, any arbitrarily small perturbation of W by some smooth localized function W_2 restores this feature, see [CMSB09, Proposition 6.4]. Nonetheless, the approximate controllability in arbitrarily small time is not possible, see [BCT14], recently extended in [BCT18]. This does not affect the mild coupling at any order as $(A, \mathrm{i}W_2)$ is also mildly coupled at any order and W_2 commutes with W which ensures that $(A, \mathrm{i}(W+W_2))$ is s-mildly coupled for every $s > 0$.

Note that existence of the dynamics is obtained in [Fuj79] for measurable in time and locally bounded in space-time control potentials. It can be extended to Radon measures controls using Section 3.2. Note that in the case of Radon measures without atoms, for instance L^1-controls, the resulting propagator is a weak solution of (1.5), see Proposition 16 and Remark 18.

A Notations and Definitions

Here T is a positive real and I an interval of $\mathbf{R}$.

Bounded operators space. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces, $L(\mathcal{X}, \mathcal{Y})$ is the space of linear bounded operator acting on $\mathcal{X}$ with values in $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$ we write $L(\mathcal{X}) := L(\mathcal{X}, \mathcal{Y})$.

Weak and strong topology. Let $(A_n)_{n\in\mathbf{N}}$ a sequence in $L(\mathcal{X}, \mathcal{Y})$, let A in $L(\mathcal{X}, \mathcal{Y})$. We say that A_n converges to A in the strong sense, or strongly, if for any ψ in $\mathcal{X}$, $(A_n\psi)_{n\in\mathbf{N}}$ converges to $A\psi$ in $\mathcal{Y}$. We say that A_n converges to A in the weak sense, or weakly, if for any ψ in $\mathcal{X}$ and ϕ in $\mathcal{Y}^*$, the topological dual of $\mathcal{Y}$, $(\phi(A_n\psi))_{n\in\mathbf{N}}$ converges to $\phi(A\psi)$ in $\mathbf{C}$.

Maximal dissipative operators on Hilbert spaces. An operator A on a Hilbert space $\mathcal{H}$ is dissipative if for any $\phi \in D(A)$, $\Re\langle\phi, A\phi\rangle \leq 0$. It is maximal dissipative if it has no proper dissipative extension.

Graph topology. Consider an operator A on a Hilbert space $\mathcal{H}$ with domain $D(A)$, the graph topology on $D(A)$ is the topology associated with the norm $\psi \in D(A) \mapsto \|\psi\|_{\mathcal{H}} + \|A\psi\|_{\mathcal{H}} \in [0, \infty)$.

Bounded variation functions. Let $E \subset \mathcal{X}$ for $\mathcal{X}$ Banach space. A family $t \in I \mapsto u(t) \in E$ is in $BV(I, E)$, i.e. is a bounded variation function from the interval I to E, if there exists $N \geq 0$ such that

$$\sum_{j=1}^{n} \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}} \leq N,$$

for any partition $(t_i)_{i=0}^n$ of I. The mapping

$$u \in BV(I, E) \mapsto \sup_{(t_i)_i} \sum_{j=1}^{n} \|u(t_j) - u(t_{j-1})\|_{\mathcal{X}}$$

is a semi-norm on $BV(I, E)$ denoted by $\mathrm{TV}(\cdot, (I, E))$ and it is called *total variation*.

The space $BV(I, E)$ endowed with the norm $\|\cdot\|_{BV(I)} := \|\cdot\|_{L^1} + \mathrm{TV}(\cdot, (I, E))$ is a Banach space.

On $BV(I, E)$, we consider the convergence of sequences given by: $(u_n)_{n\in\mathbf{N}} \in BV(I, E)$ *converges* to $u \in BV(I, E)$ if $(u_n)_{n\in N}$ is a bounded sequence in $BV(I, E)$ pointwise convergent to $u \in BV(I, E)$.

Notice that the convergence in the norm $\|\cdot\|_{BV(I)}$ implies pointwise convergence.

The Jordan Decomposition Theorem provides that any bounded variation function is the difference of two nondecreasing bounded functions. This fact, together with Helly's Theorem provides the well-known Helly's Selection Theorem (see for example [Hel12, Nat55]).

Theorem (Helly's Selection Theorem)**.** Let I be a compact interval and $(f_n)_{n\in\mathbf{N}}$ be a sequence in $BV(I, \mathbf{R})$. If

(*i*) there exists $M > 0$ such that for all $n \in \mathbf{N}$, $\mathrm{TV}(f_n, (I, \mathbf{R})) < M$,

(*ii*) there exists $x_0 \in I$ such that $(f_n(x_0))_{n\in\mathbf{N}}$ is bounded.

Then $(f_n)_{n\in\mathbf{N}}$ has a pointwise convergent subsequence.

Radon measures. We consider the space $\mathcal{R}(I)$ of (signed) Radon measures on I. Recall that a positive Radon measure is a Borel measure which is locally finite and inner regular. Using Hahn decomposition [Dos80] any signed Radon measure μ is the difference $\mu = \mu^+ - \mu^-$ of two positive Radon measures μ^+ and μ^- (at least one being finite) with disjoint support. We denote the total variation of μ by $|\mu|(I)$, where $|\mu| = \mu^+ + \mu^-$. In this work we consider Radon measures with bounded total variation. In particular both μ^+ and μ^- are finite.

Here we only consider finite measures on I so the inner regularity requirement in the definition can be dropped. In the more general σ-finite case, this requirement can be dropped as well. In the first case, a positive Radon measures is a finite Borel measures, while in the second case a positive Radon measure is a locally finite Borel measure. Note that, sometimes Borel measures are by definition locally finite. Sometimes the outer regularity is added to the definition of Radon measures, which again is redundant for finite measures.

We say that $(\mu_n)_{n\in\mathbf{N}} \in \mathcal{R}([0,T])$ *converges* to $\mu \in \mathcal{R}([0,T])$ if $\sup_n |\mu_n|([0,T]) < +\infty$ (i.e. $(\mu_n)_{n\in\mathbf{N}}$ has uniformly bounded total variations) and $\mu_n((0,t]) \to \mu((0,t])$ for every $t \in (0,T]$ as n tends to ∞. Note that this convergence is *not* the one associated with the norm of total variation, see also Remark 13. Notice, moreover, that this notion of convergence is stronger than the weak convergence of measures, see [EG92, Section 1.9] and weaker than the strong or total variation convergence. It is also stronger than the narrow convergence (also called weak convergence in [Bil99, Kle14, Mat95]). For instance, the sequence $\left(\delta_{\frac{1}{n}}\right)_{n\in\mathbf{N}}$ converges narrowly to δ_0 but is not convergent according to our definition.

The cumulative function $u(t) = \mu((0,t])$ of a Radon measure μ is locally of bounded variation and the associated total variation (which does not depend on the choice of the cumulative function) coincides with the total variation of the Radon measure.

Every function $u \in L^1_{\text{loc}}(I,\mathbf{R})$ can be seen as the density of an absolutely continuous Radon measure μ, namely $\mu(J) = \int_J u \mathrm{d}\lambda$, (where λ denotes the Lebesgue measure) for every $J \subset I$ borelian. When it does not create ambiguity we identify the function u and the associated Radon measure μ. Moreover we have the following convergence.

Lemma. Let $(u_n)_{n\in\mathbf{N}} \subset L^1(I,\mathbf{R})$ and $u \in L^1(I,\mathbf{R})$ such that $u_n \to u$ in $L^1(I,\mathbf{R})$ as n tends to ∞. Let $(\mu_n)_{n\in\mathbf{N}} \subset \mathcal{R}(I)$ and $\mu \in \mathcal{R}(I)$ be the associated Radon measures. Then $(\mu_n)_{n\in\mathbf{N}}$ converges to μ in $\mathcal{R}(I)$.

Note that for u in $L^1(I,\mathbf{R})$ the total variation of the associated Radon measure is the L^1-norm of u and hence $L^1(I,\mathbf{R})$ is closed for the total variation topology.

Other notations. For any interval $I \subset \mathbf{R}$, we define

$$\Delta_I := \{(s,t) \in I^2 \mid s \leq t\}.$$

In a metric space E, the notation $B_E(v_0, r)$ stands for the open ball of radius r and center v_0 in E. For a densely defined operator B on a Hilbert space, B^* stands for its adjoint. Recall that B^* is densely defined if and only if B is closable, in any case B^* is closed.
The set $C^1_0(I,\mathcal{X})$ is the set of of functions from an interval I to a Banach $\mathcal{X}$ of class C^1 with compact support in the interior of I.

B Interpolation

B.1 Convergence of sequences

Through the present analysis, the following simple interpolation lemma is useful.

Lemma 38. *Let A be a skew-adjoint operator, let S be a set and $(u_n)_{n\in\mathbf{N}}$ take value in the set of functions from S to $D(|A|^k)$, such that $(u_n)_{n\in N}$ is uniformly bounded in S for the norm of $D(|A|^k)$, $k>0$. If $(u_n)_{n\in\mathbf{N}}$ tends to zero in $\mathcal{H}$ uniformly in S, then $(u_n)_{n\in\mathbf{N}}$ tends to zero in $D(|A|^l)$, uniformly in S for every $l<k$.*

Proof. The proof follows from the logarithmic convexity of $l\in[0,k]\mapsto \||A|^l u\|$. Indeed

$$\||A|^{\frac{l+j}{2}}u\| = \sqrt{\langle |A|^l u, |A|^j u\rangle} \leq \||A|^l u\|^{1/2}\||A|^j u\|^{1/2}.$$

If $l<k$ then

$$\||A|^l u_n\| \leq \|u_n\|^{\frac{k-l}{k}}\||A|^k u_n\|^{\frac{l}{k}}.$$

Let $C=\sup_{n\in\mathbf{N}}\||A|^k u_n\|^2$ and $N>0$ such that for any $n>N$, $\|u_n\|^2\leq\varepsilon$ we obtain

$$n>N \Longrightarrow \||A|^l u_n\|^2 \leq \varepsilon^{\frac{k-l}{k}} C^{\frac{l}{k}},$$

which provides the lemma. □

B.2 Interpolation of fractional powers of operators

Let us now state a more sophisticated result. The following result can also be deduced from the content of [ABG96, Section 2.8] and its proof is an extension to the unbounded case of the result by [Ped72].

Proposition 39. *Let A and B be two self-adjoint positive operators in $\mathcal{H}$ such that there exists $c>0$ with*

$$c\leq B\leq A$$

in the form sense. Then, for any $\alpha\in(0,1)$,

$$c^\alpha \leq B^\alpha \leq A^\alpha.$$

The proof of Proposition 39 follows from the following series of lemmas.

For a selfadjoint operator A and $z\in\mathbf{C}\setminus\mathbf{R}$, the functional calculus is the extension of the mapping

$$\left\{x\in\mathbf{R}\mapsto (x-z)^{-1}\right\}\in B(\mathbf{R}) \to (A-z)^{-1}\in B(\mathcal{H})$$

as a strong continuous $*$-algebra homomorphism on the space $B(\mathbf{R})$ of bounded borelian functions on the real line to $B(\mathcal{H})$.

Let us recall the following functional calculus identity based on the Poisson formula, see [ABG96, Lemma 6.1.1].

Lemma 40. *Let A be a selfadjoint operator in $\mathcal{H}$. Let f be a bounded borelian function. Then $f(A)$ is the weak-limit as $\varepsilon\to 0^+$ of*

$$\frac{1}{2i\pi}\int_{\mathbf{R}} f(\lambda)\Im(A-\lambda-i\varepsilon)^{-1}\,d\lambda.$$

We also recall, for $\alpha\in(0,1)$ and $x>0$, the formula

$$x^{-\alpha} = \frac{\pi}{\sin(\pi\alpha)}\int_0^\infty \frac{w^{-\alpha}}{x+w}\,dw.$$

Then from the Fubini theorem and Lemma 40 we obtain the following.

Lemma 41. *Let A be a positive selfadjoint operator in $\mathcal{H}$. Then for $\alpha \in (0,1)$*

$$A^\alpha = \frac{\pi}{\sin(\pi\alpha)} \int_0^\infty \frac{w^{-1+\alpha} A}{A+w}\, dw$$

on $D(A)$.

The domain of validity of the above identity can be extended to any core of A^α that makes the integral strongly convergent.

Lemma 42. *Let A and B be two self-adjoint positive operators in $\mathcal{H}$ such that there exists $c > 0$ with*

$$c \leq B \leq A.$$

Then

$$A^{-1} \leq B^{-1}.$$

Proof. First notice that both A and B are invertible from their domains to $\mathcal{H}$ as well as their square roots. Then from

$$\sqrt{c}\|u\| \leq \|\sqrt{B}u\| \leq \|\sqrt{A}u\|,$$

we deduce that $\sqrt{B}\sqrt{A}^{-1}$ is a bounded operator with norm at most 1.

In the other hand the operator $\sqrt{A}^{-1}\sqrt{B}$ defined on $D(\sqrt{B})$ extends as the adjoint of $\sqrt{B}\sqrt{A}^{-1}$ to a closed operator on $\mathcal{H}$ and hence is bounded with norm at most 1 and

$$\|\sqrt{A}^{-1}\sqrt{B}u\| \leq \|u\|, \forall u \in D(\sqrt{B})$$

and thus

$$\|\sqrt{A}^{-1}u\| \leq \|\sqrt{B}^{-1}u\|.$$

and the result follows. $\square$

Proof of Proposition 39. We have that

$$c \leq B \leq A,$$

which implies, for any $w > 0$, that

$$1 - w(B+w)^{-1} \leq 1 - w(A+w)^{-1}.$$

and thus

$$\frac{w^{-1+\alpha}B}{B+w} \leq \frac{w^{-1+\alpha}A}{A+w}.$$

Integrating on $w > 0$ (first restricted to $D(A) \times D(A)$) gives the desired inequality by density. $\square$

Proposition 39 can be extend to the case $c = 0$ by replacing A and B by $A + \epsilon$ and $B + \epsilon$ as in [Ped72], we then obtain

$$0 \leq B^\alpha \leq (B+\epsilon)^\alpha \leq (A+\epsilon)^\alpha.$$

The second inequality is immediate. We have that

$$0 \leq (A+\epsilon)^{-\alpha/2} B^\alpha (A+\epsilon)^{-\alpha/2} \leq 1,$$

so that, taking ϵ to 0, gives

$$0 \leq B^\alpha \leq A^\alpha. \tag{B.1}$$

We hence deduce the following corollary.

Corollary 43. *Let A and B be two positive self-adjoint operators sharing the same domains. For any $\alpha \in (0,1)$, we have :*

$$D(A^\alpha) = D(B^\alpha)$$

Proof. As B is closed it is a bounded operator from $D(A)$ to $\mathcal{H}$. Thus

$$\exists c > 0, \forall \phi \in D(A), \|B\phi\| \leq c\|A\phi\|.$$

Hence

$$B^2 \leq c^2 A^2,$$

and, from (B.1), we have that $B^{2\alpha/2}$ is bounded from $D(A^{2\alpha/2})$ to $\mathcal{H}$. We conclude by noticing that the proof is symmetric in A and B. $\square$

C Sufficient conditions for approximate controllability with bounded variation controls

The aim of this Section is to recall approximate controllability results obtained in other contexts and how this results may be adapted to the framework of the present analysis.

We first recall the following definitions from [CMSB09].

Definition 7. Let $(A, B, \mathbf{R})$ satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of $\mathcal{H}$ made of eigenvectors of A, $A\phi_k = \mathrm{i}\lambda_k\phi_k$ for every k in $\mathbf{N}$. A pair (j,k) of integers is a *nondegenerate transition* of (A, B, Φ) if (i) $\langle \phi_j, B\phi_k \rangle \neq 0$ and (ii) for every (l,m) in $\mathbf{N}^2$, $|\lambda_j - \lambda_k| = |\lambda_l - \lambda_m|$ implies $(j,k) = (l,m)$ or $\langle \phi_l, B\phi_m \rangle = 0$ or $\{j,k\} \cap \{l,m\} = \emptyset$.

Definition 8. Let $(A, B, \mathbf{R})$ satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of $\mathcal{H}$ made of eigenvectors of A, $A\phi_k = \mathrm{i}\lambda_k\phi_k$ for every k in $\mathbf{N}$. A subset S of $\mathbf{N}^2$ is a *nondegenerate chain of connectedness* of (A, B, Φ) if (i) for every (j,k) in S, (j,k) is a nondegenerate transition of (A, B) and (ii) for every r_a, r_b in $\mathbf{N}$, there exists a finite sequence $r_a = r_0, r_1, \ldots, r_p = r_b$ in $\mathbf{N}$ such that, for every $j \leq p-1$, (r_j, r_{j+1}) belongs to S.

Proposition 44. *Let $(A, B, \mathbf{R})$ satisfy Assumptions 1 such that A and B are skew-symmetric. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of $\mathcal{H}$ made of eigenvectors of A, $A\phi_k = \mathrm{i}\lambda_k\phi_k$ for every k in $\mathbf{N}$. Let S be a nondegenerate chain of connectedness of (A, B). Then, for every $\eta > 0$, (A, B) is simultaneously approximately controllable in $D(|A|^{1-\eta})$.*

Proof. First of all, it is enough to prove the result for target propagators $\hat{\Upsilon}$ leaving invariant the space of co-dimension 2 spanned by (ϕ_j, ϕ_k) for (j,k) in S

$$\hat{\Upsilon} = e^{\mathrm{i}\nu_l}(\cos(\theta)\phi_l^*\phi_l + \sin(\theta)\phi_l^*\phi_k) + e^{\mathrm{i}\nu_k}(-\sin(\theta)\phi_k^*\phi_l + \cos(\theta)\phi_l^*\phi_k).$$

The result in $\mathcal{H}$-norm is a consequence of [Cha12, Theorem 1]: for every piecewise constant $u^* : \mathbf{R} \to \mathbf{R}$, $2\pi/|\lambda_j - \lambda_k|$-periodic such that

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{\mathrm{i}(\lambda_j - \lambda_k)\tau} \mathrm{d}\tau \neq 0,$$

and

$$\int_0^{\frac{2\pi}{|\lambda_j - \lambda_k|}} u^*(\tau) e^{\mathrm{i}(\lambda_l - \lambda_m)\tau} \mathrm{d}\tau = 0,$$

for every l, m such that $(\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k)$ and $b_{l,m} \neq 0$, there exists T^* such that $\Upsilon^{u^*/n}(nT^*, 0)$ tends to $\hat{\Upsilon}$ as n tends to infinity.

The conclusion follows using Lemma 38 and the estimate in A-norm of Theorem 5. $\square$

Let us just mention the following result in the case of higher regularity.

Proposition 45. *Let k be a positive real. Let $(A, B, \mathbf{R})$ satisfy Assumptions 1 such that (A, B) is k-mildly coupled. Let $\Phi = (\phi_k)_k$ be a Hilbert basis of $\mathcal{H}$ made of eigenvectors of A, $A\phi_k = \mathrm{i}\lambda_k\phi_k$ for every k in $\mathbf{N}$. Let S be a nondegenerate chain of connectedness of (A, B) such that, for every (j, k) in S, the set $\{(l, m) \in \mathbf{N}^2 | (\lambda_l - \lambda_m) \in \mathbf{Z}(\lambda_j - \lambda_k)$ and $\langle\phi_l, B\phi_m\rangle \neq 0\}$ is finite. Then, for every $\eta > 0$, (A, B) is simultaneously approximately controllable in $D(|A|^{k/2+1-\eta})$.*

Proof. The proof differs from the previous one for the interpolation step and for the use of Proposition 25. □

D Analytical perturbations

To apply sufficient condition for approximate controllability (Proposition 45), we need to find a nonresonant chain of connectedness, which may require some work on practical examples. A classical idea already used in this study is to introduce a new control $\tilde{u} = u - \bar{u}$ and to consider the system $x' = (A + \bar{u}B) + (u - \bar{u})B$ for a suitably chosen constant $\bar{u}$.

We have the following results by Kato [Kat66, Section VII.2].

Definition 9. Let D_0 be a domain of the complex plane, a family $(T(z))_{z\in D_0}$ of closed operators from a Banach space X to a Banach space Y is said to be *a holomorphic family of type (A)* if

1. $D(T(z)) = D$ is independent of z,
2. $T(z)u$ is holomorphic for z in D_0 for every u in D.

Theorem 46 ([Kat66, Theorem VII.3.9])**.** *Let $T(z)$ be a* selfadjoint *holomorphic family of type (A) defined for z in a neighborhood of an interval I_0 of the real axis such that $T(z)^* = T(\bar{z})$. Furthermore, let $T(z)$ have a compact resolvent. Then all eigenvalues of $T(z)$ can be represented by functions which are holomorphic in I_0*[1]*.*

More precisely, there is a sequence of scalar-valued functions $(z \mapsto \lambda_n(z))_{n\in\mathbf{N}}$ and operator-valued functions $(z \mapsto \phi_n(z))_{n\in\mathbf{N}}$, all holomorphic on I_0, such that for z in I_0, the sequence $(\lambda_n(z))_{n\in\mathbf{N}}$ represents all the repeated eigenvalues of $T(z)$ and $(\phi_n(z))_{n\in\mathbf{N}}$ forms a complete orthonormal family of the associated eigenvectors of $T(z)$.

Proposition 47. *If (A, B, K) satisfies Assumptions 1 then the family $\mathrm{i}(A + zB)_{z\in\mathbf{C},|z|<1/\|B\|_A}$ is holomorphic of type (A).*

Proof. The question of domain is solved by the Kato–Rellich Theorem. The holomorphy is immediate as the family $\mathrm{i}(A + zB)$ is affine in z. □

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